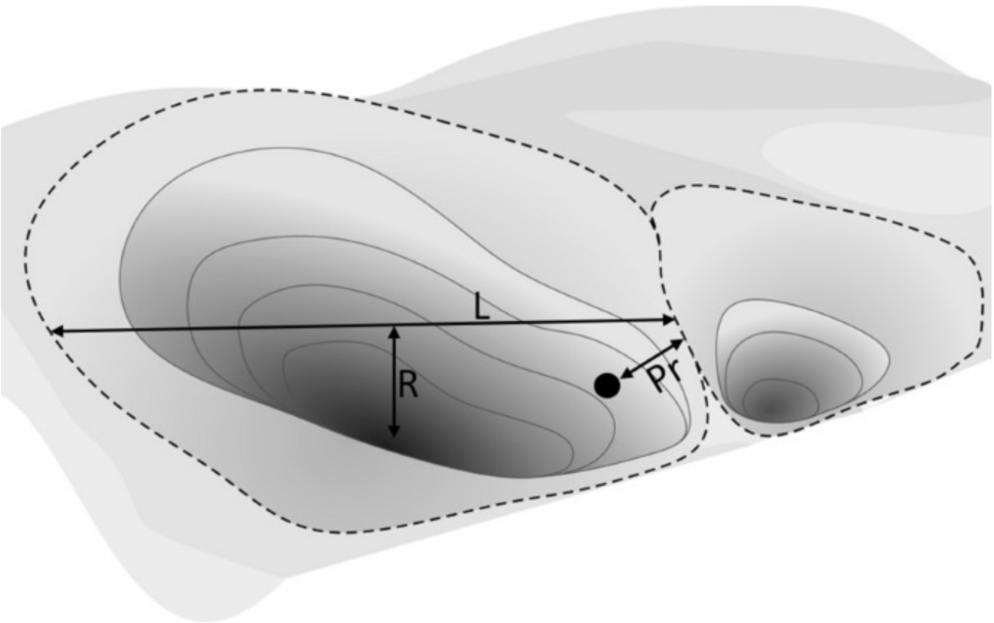
# Mathematics of Resilience
## with Applications to Climate



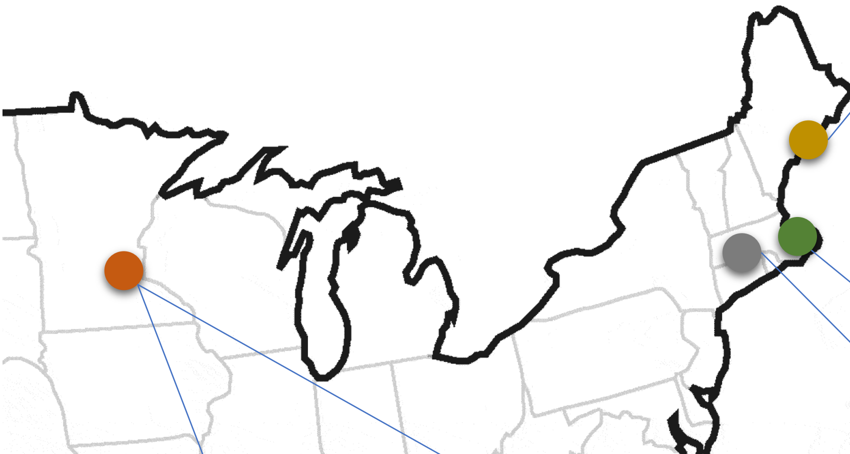

**Kate Meyer, Carleton College**

Guest Lecture, UMN Math 5490

September 28, 2023

Image: https://link.springer.com/article/10.1007/s10021-021-00737-2

# Collaborators



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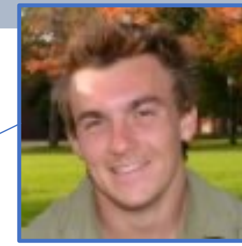
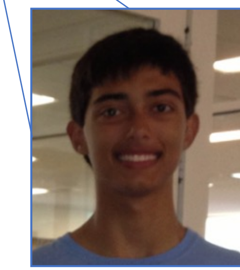
Bowdoin Coll.



**Stephen Ligtenberg**



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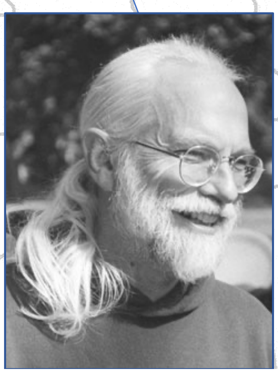

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Mt. Holyoke Coll.

**Sarah Iams**
Harvard U.

# Math for environmental decision support?

# Math for environmental decision support?

Resilience quantification

Part 1: Resilience frameworks

Part 2: Flow-kick models for quantifying resilience to repeated disturbances

Part 3: Climate applications

# What does Resilience Mean to You?

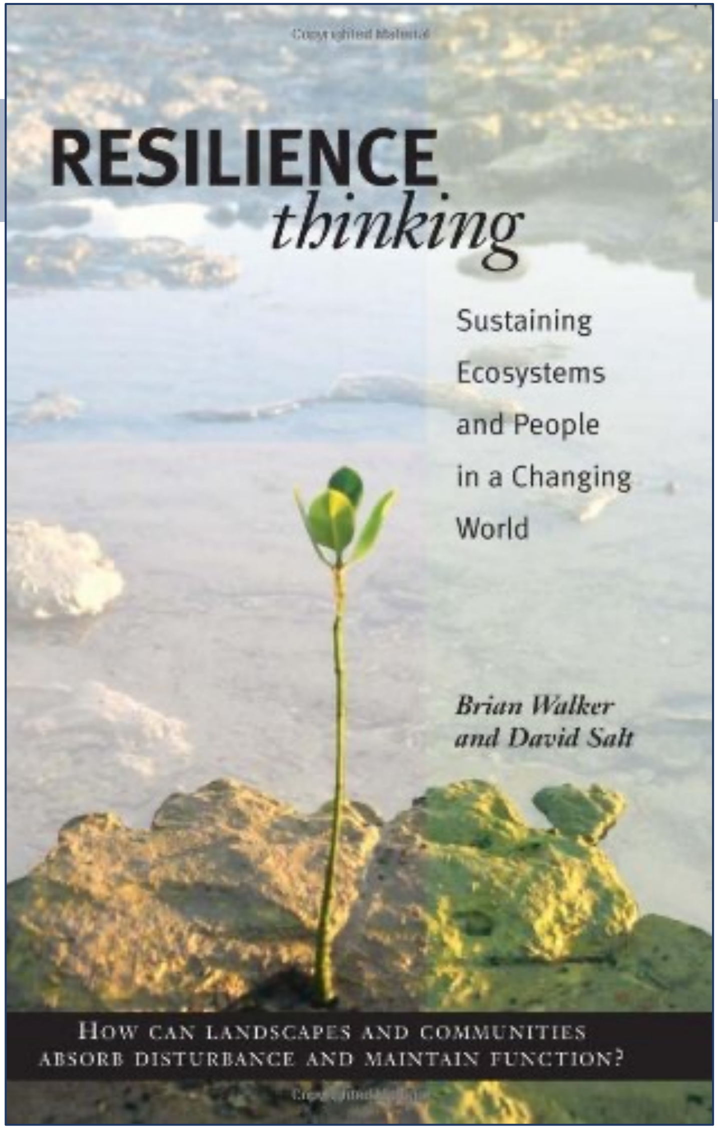
THINK – PAIR – SHARE

- Being able to **bounce back** from setbacks, learn and grow
- Getting pushed---at what point does it push too far?
- Withstand change from external influence
- Adapt to change

# One take on Resilience:

*"[T]he **capacity** of [a] system to **absorb change** and disturbances and still **retain** its basic **structure and function**"*

*- Brian Walker and David Salt*



Ecosystems (2001) 4: 765–781
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ECOSYSTEMS
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MINIREVIEW

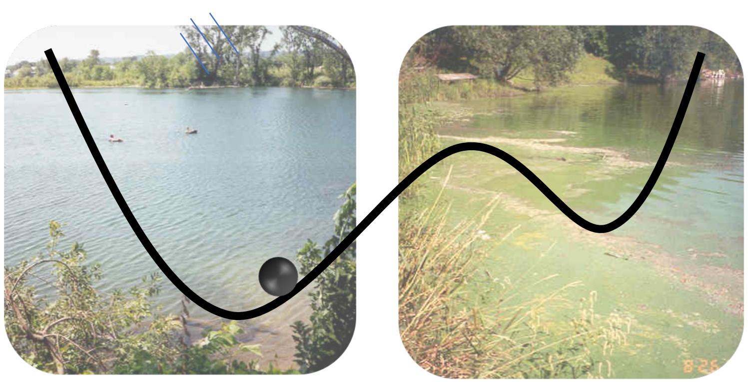
# From Metaphor to Measurement: Resilience of What to What?

Steve Carpenter,[1]* Brian Walker,[2] J. Marty Anderies,[2] and Nick Abel[2]

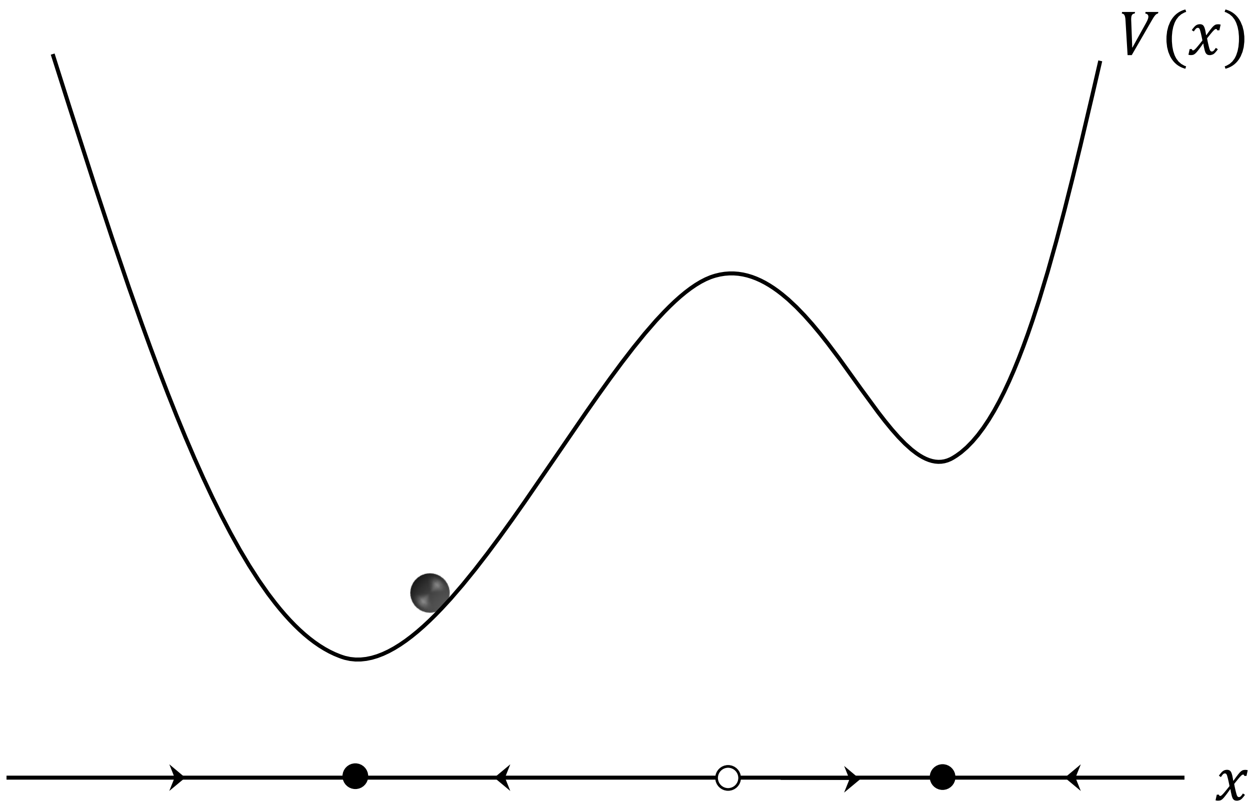
**structure and function of system:**
- **basin of attraction (feedback regime)**
- value derived from system

**disturbance type:**
- small / medium / large
- one-time / repeated
- continuous / discrete

...

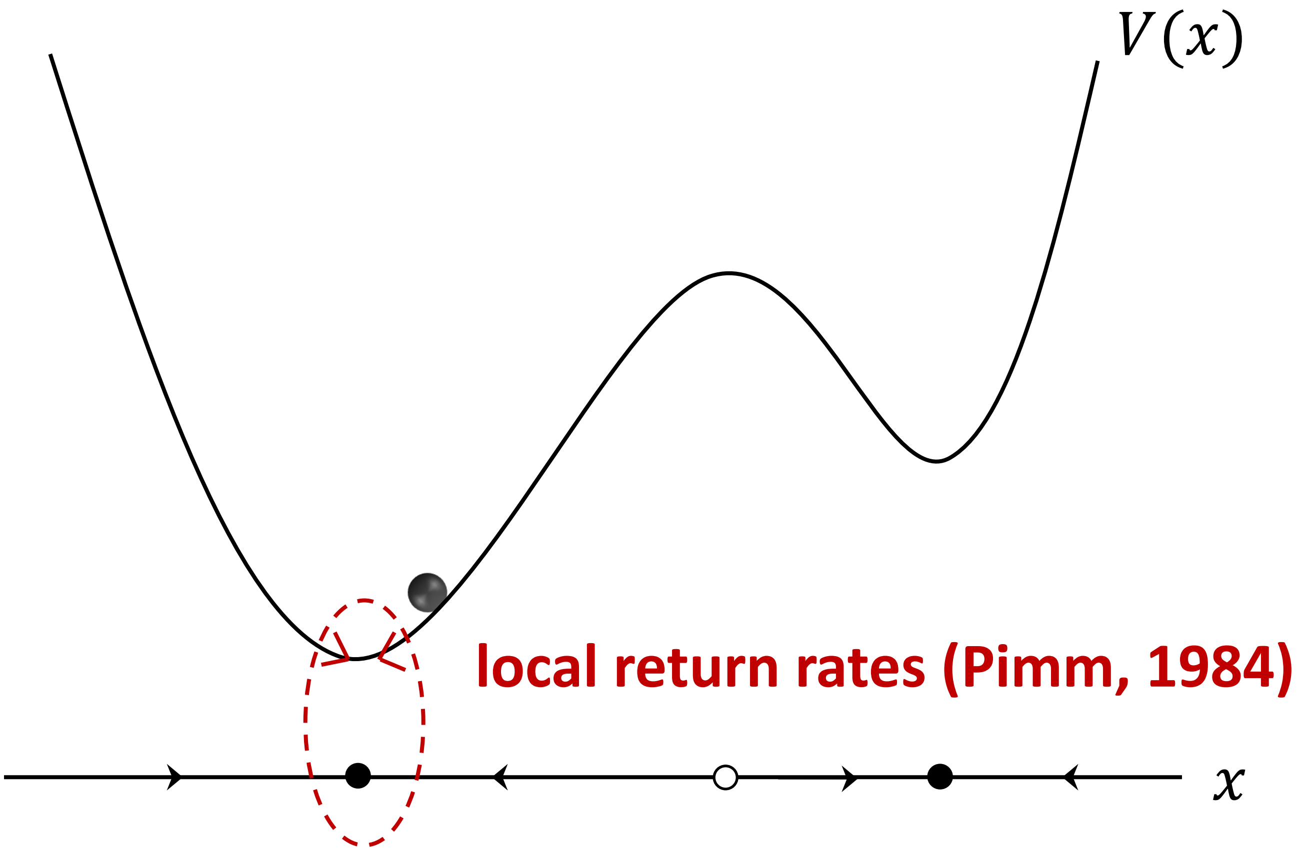


# Resilience metrics

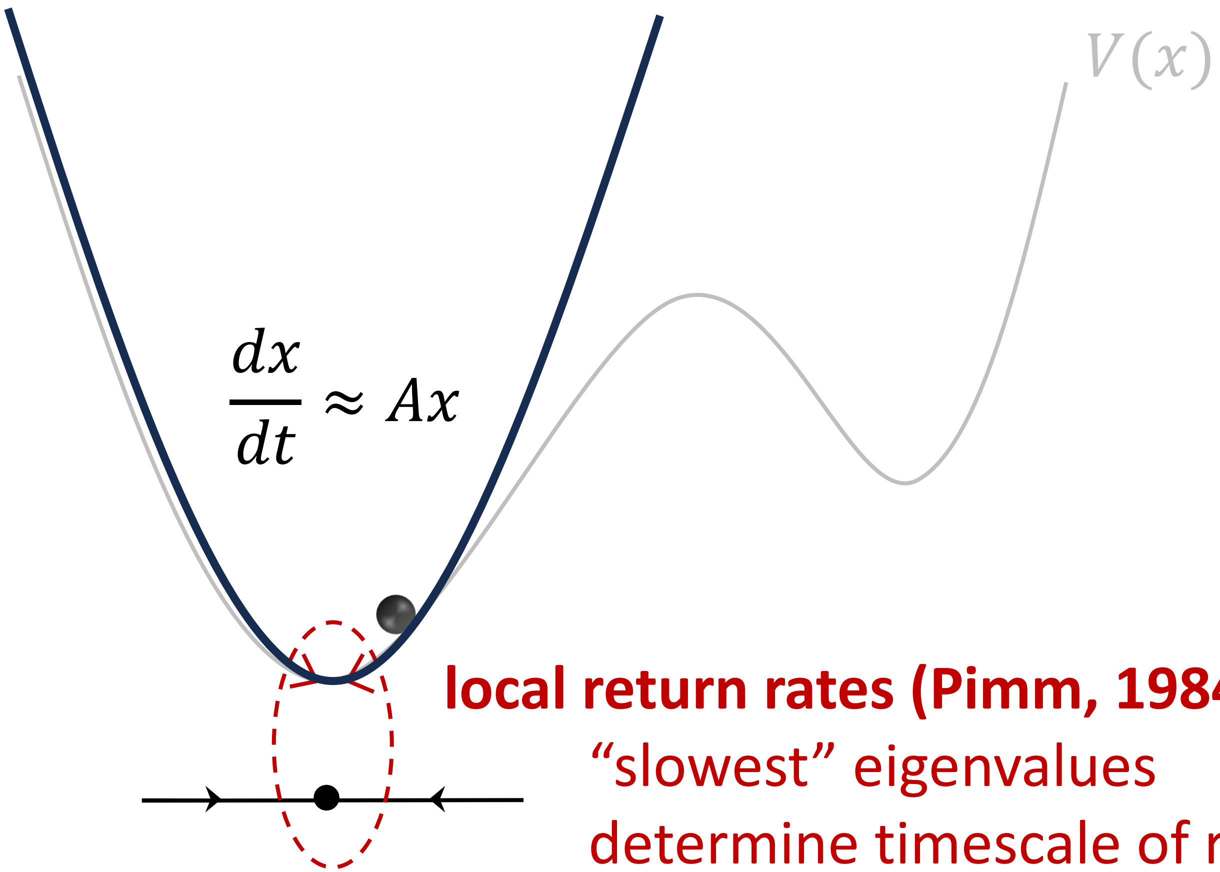



$$\frac{dx}{dt} = f(x) = -V'(x)$$

# Resilience metrics



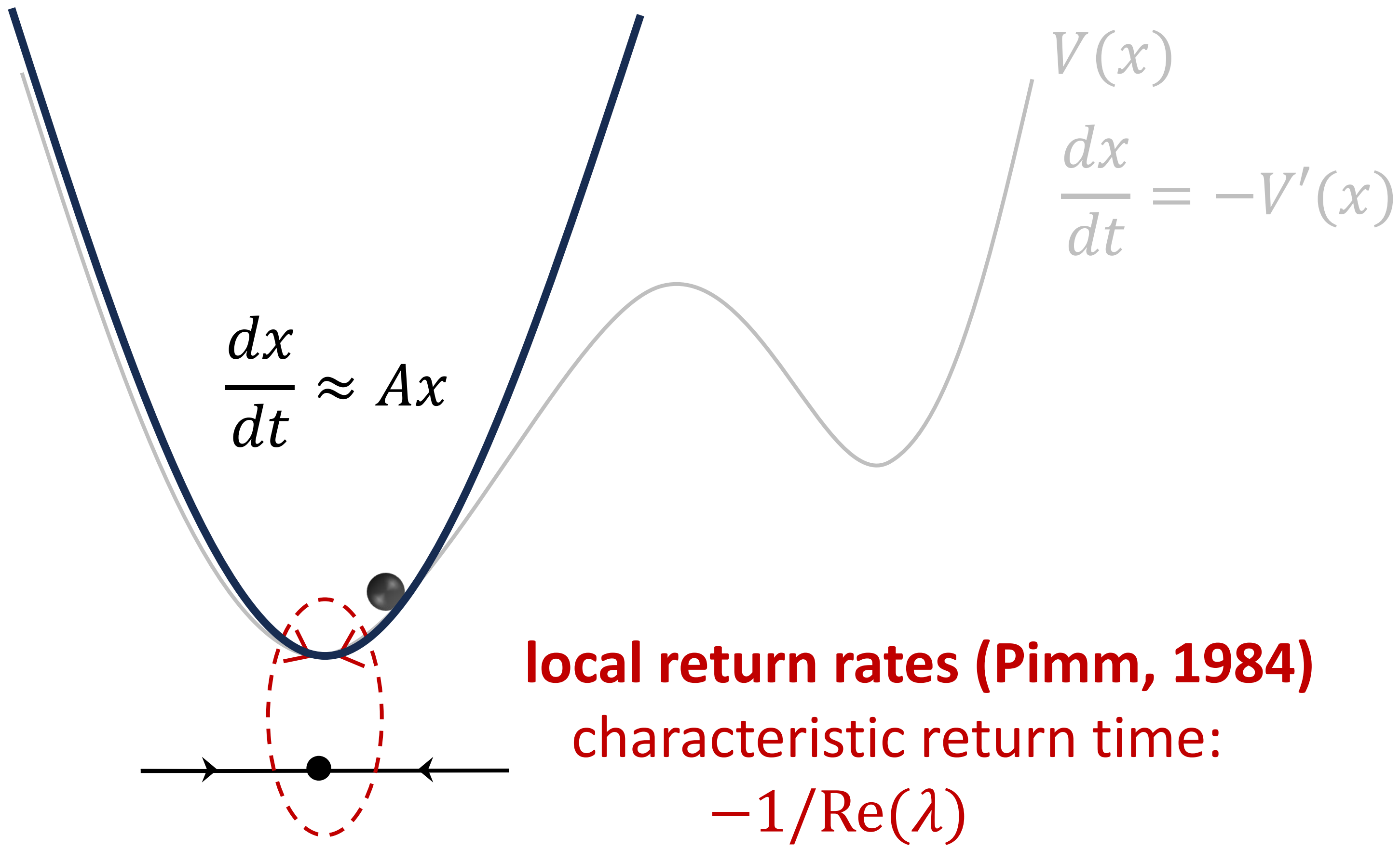
# Resilience metrics




**local return rates (Pimm, 1984)**

"slowest" eigenvalues determine timescale of recovery from perturbation

# Resilience metrics



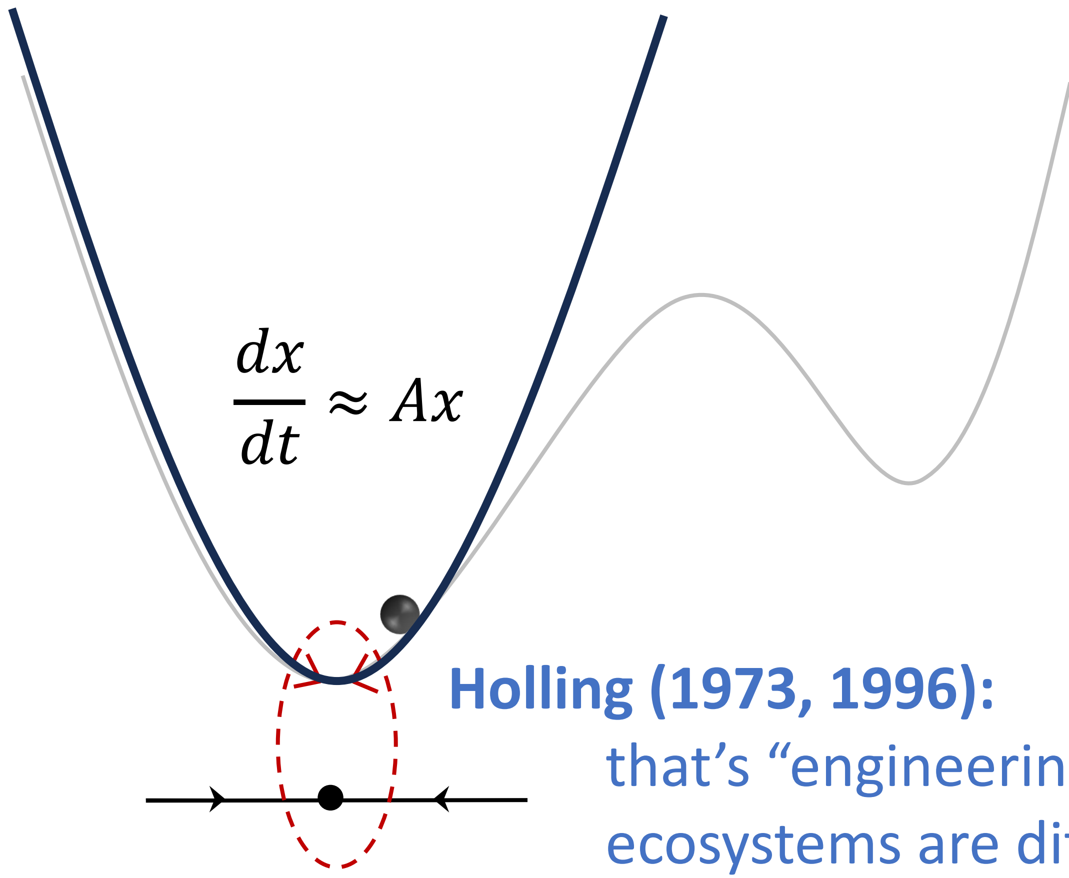

**local return rates (Pimm, 1984)**

characteristic return time:

$-1/\mathrm{Re}(\lambda)$

weakest eigenvalue of $A$

# Resilience metrics



$$V(x)$$

$$\frac{dx}{dt} = -V'(x)$$

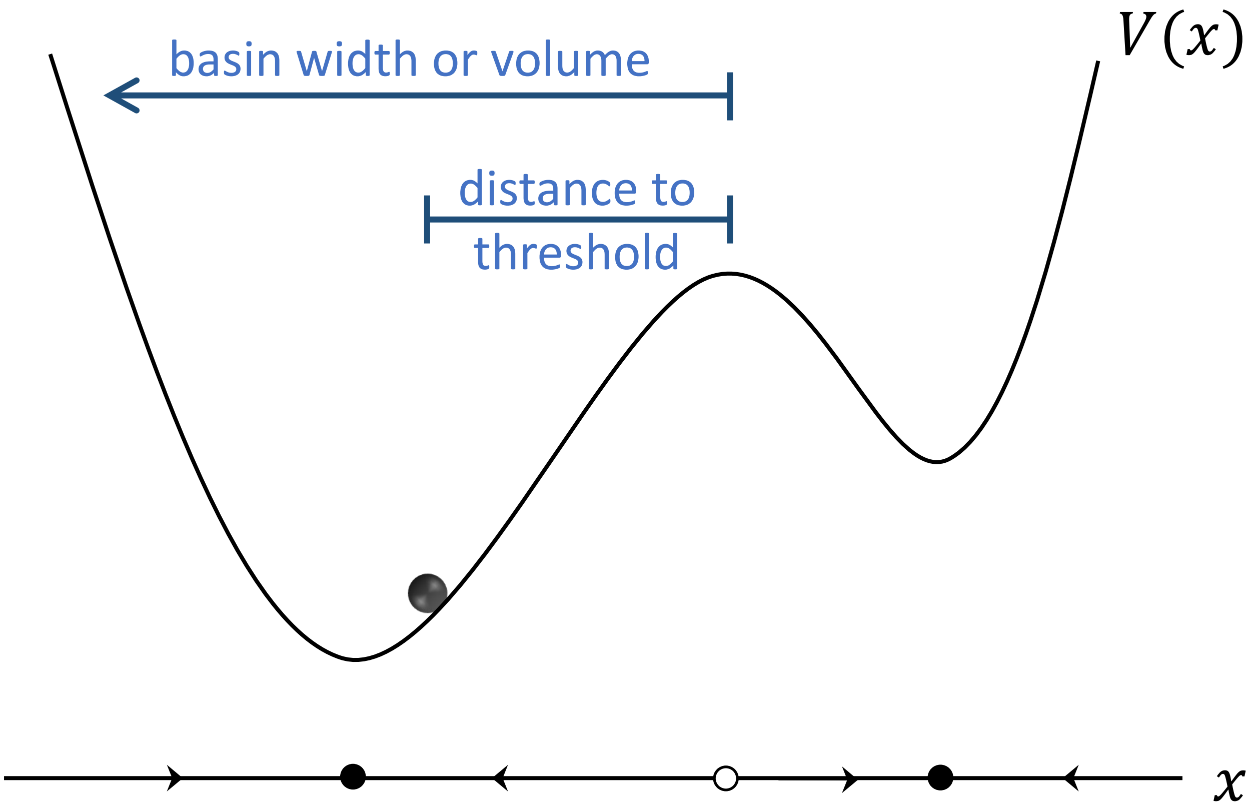
$$\frac{dx}{dt} \approx Ax$$

**Holling (1973, 1996):**

that's "engineering resilience".
ecosystems are different than airplanes!

# Resilience metrics

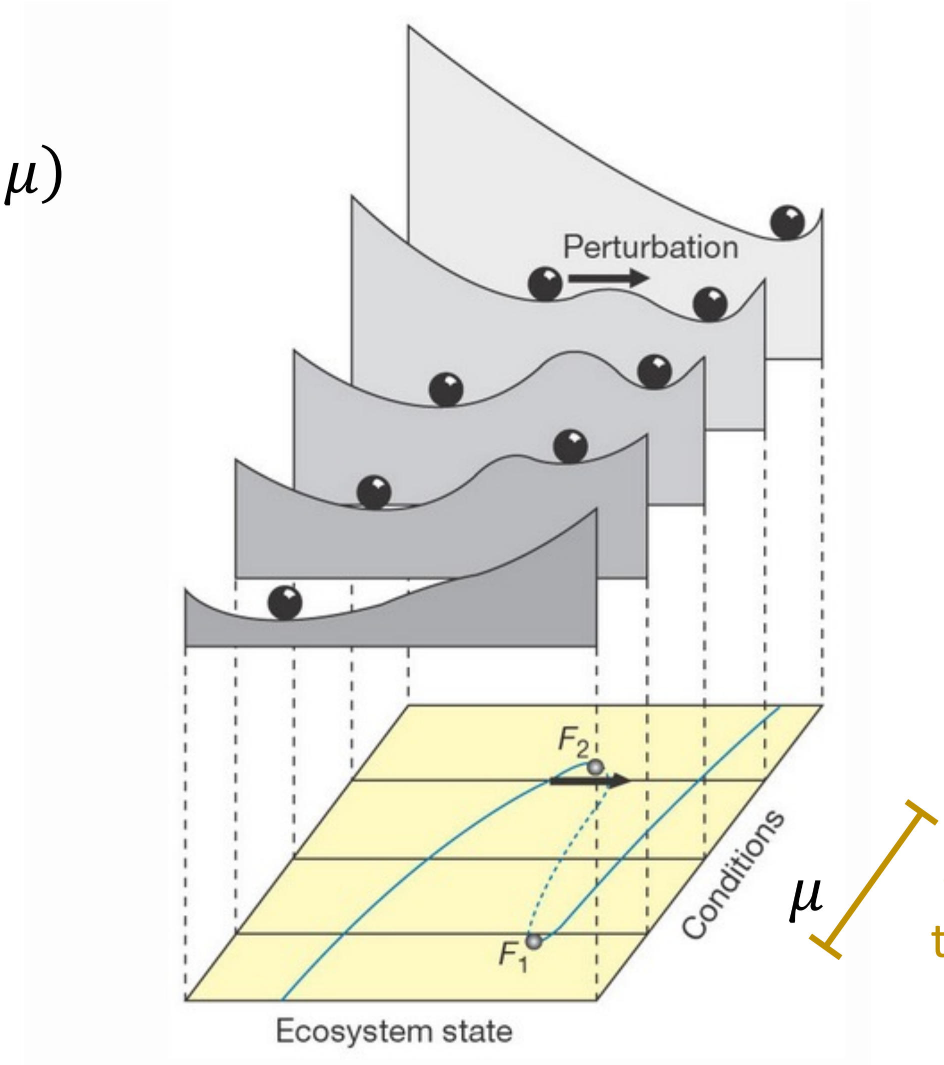
Reviewed in Meyer 2016, *Nat. Resour. Model.* 29(3)




$$\frac{dx}{dt} = f(x) = -V'(x)$$

# Resilience metrics

$$\frac{dx}{dt} = f(x, \mu)$$

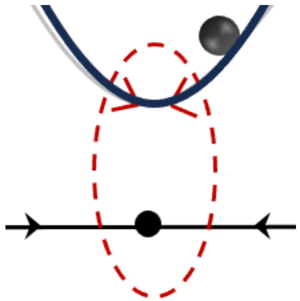



distance to threshold in parameter space

Scheffer et al., 2001, *Nature* 413

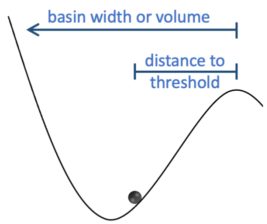
# Resilience of What to What?

measures resilience of attracting state to…



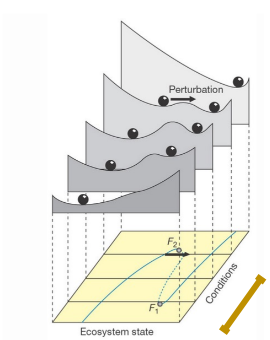
local
return rates

**small infrequent**, discrete dist.




distance to threshold in state ($x$) space

large disturbance (exceed basin width?)




distance to threshold in parameter space

continuous disturbances (reshape the stability landscapes); changing environmental factors

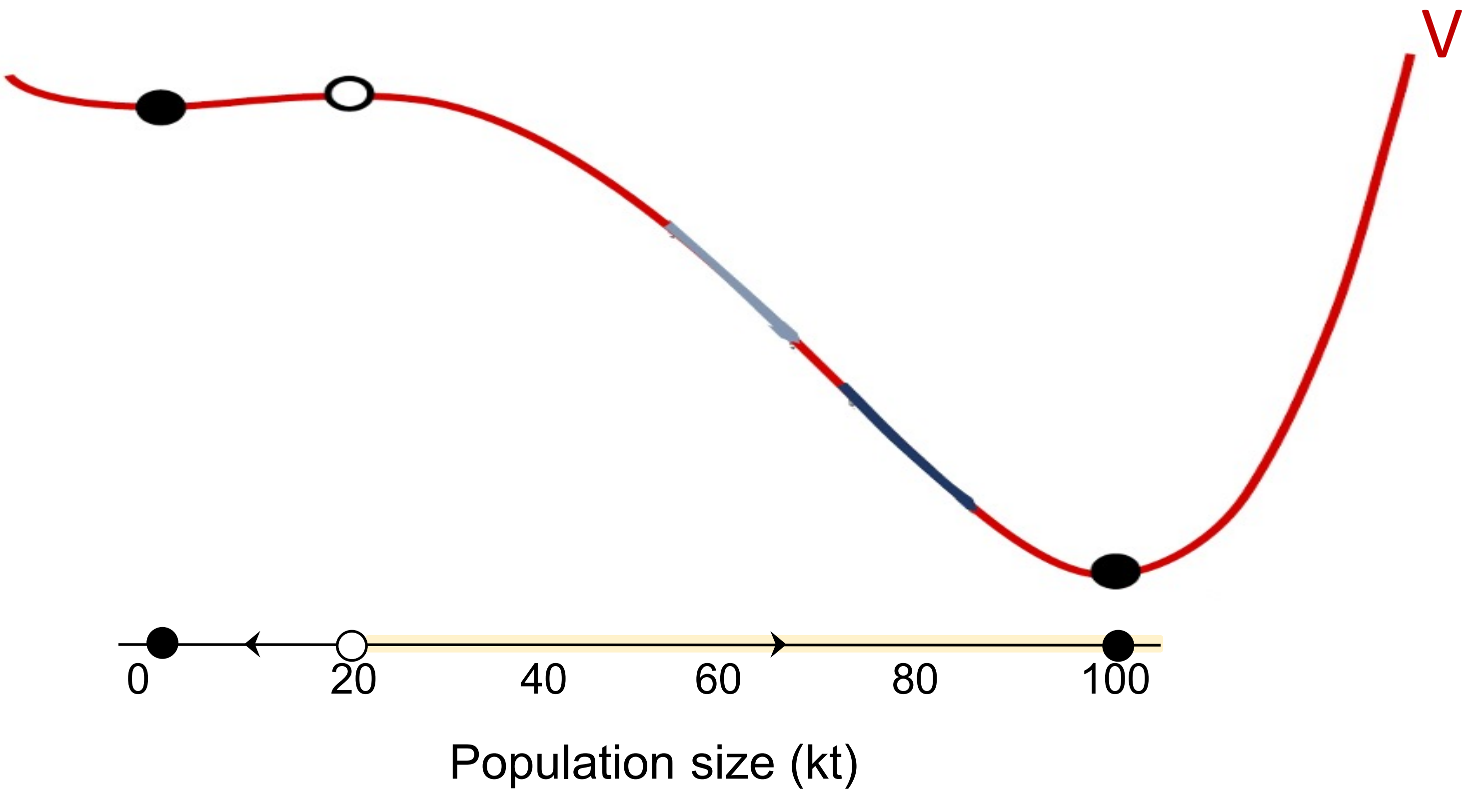
# Resilience of What to What?

?

resilience of an attracting state to ***repeated*** perturbations

# Example: fish population

Meyer et al. 2018, *Nat. Sustain.* 1



# Example: fish population

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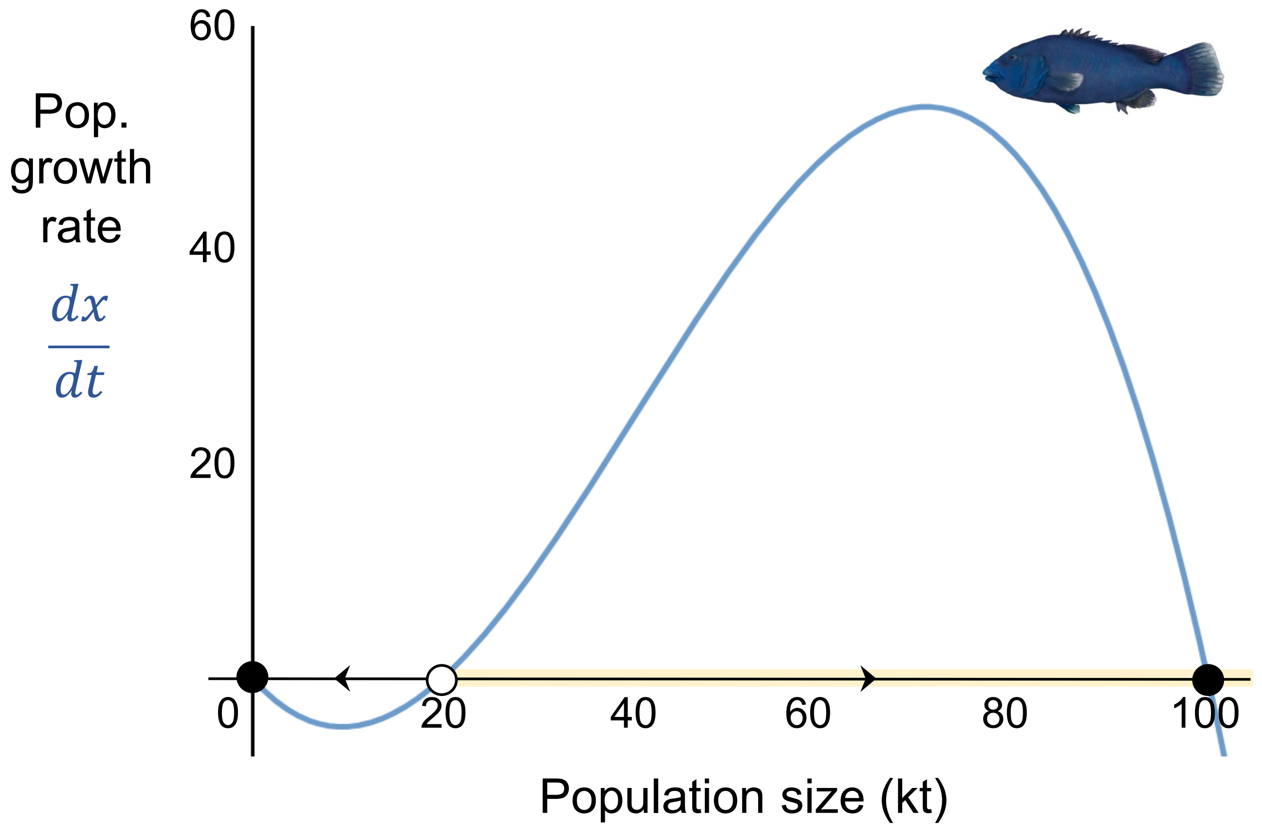
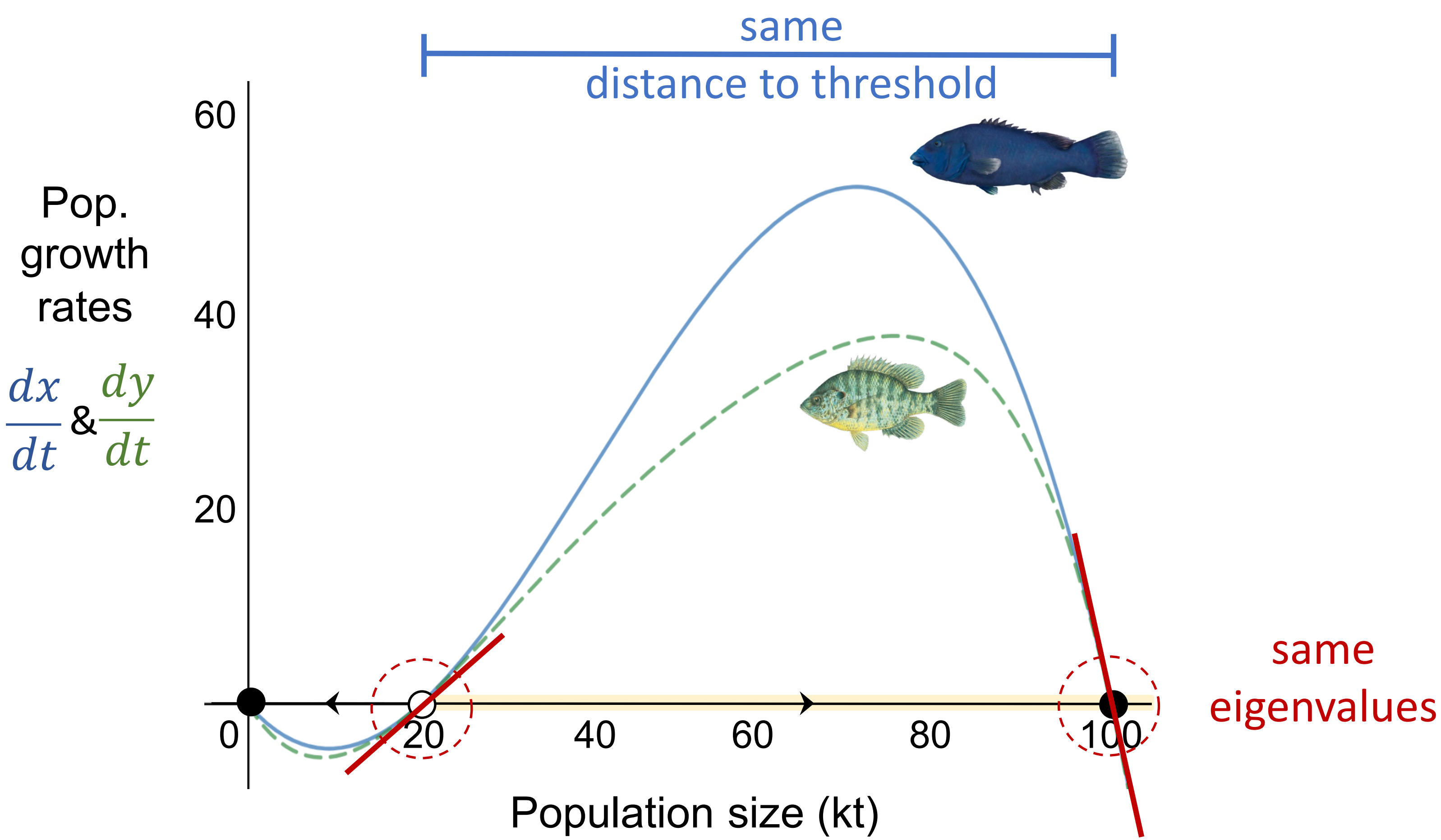



$$\text{population growth rate} = \frac{dx}{dt} = x\left(1 - \frac{x}{100}\right)\left(\frac{x}{20} - 1\right)$$

# Two fish populations




$$\frac{dx}{dt} = x\left(1 - \frac{x}{100}\right)\left(\frac{x}{20} - 1\right)$$

$$\frac{dy}{dt} = y\left(1 - \frac{y}{100}\right)\left(\frac{y}{20} - 1\right) \cdot (0.0002y^2 - 0.024y + 1.4)$$

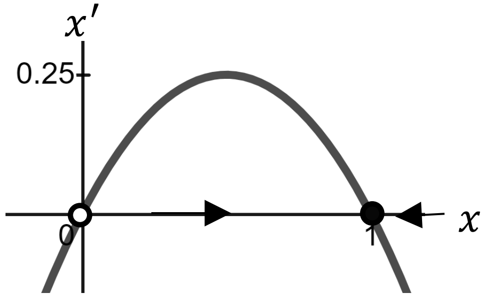
Part 1: Resilience frameworks

Part 2: Flow-kick models for quantifying resilience to repeated disturbances

Part 3: Climate applications

# Flow example: logistic growth

$$\frac{dx}{dt} = f(x) = x(1 - x)$$



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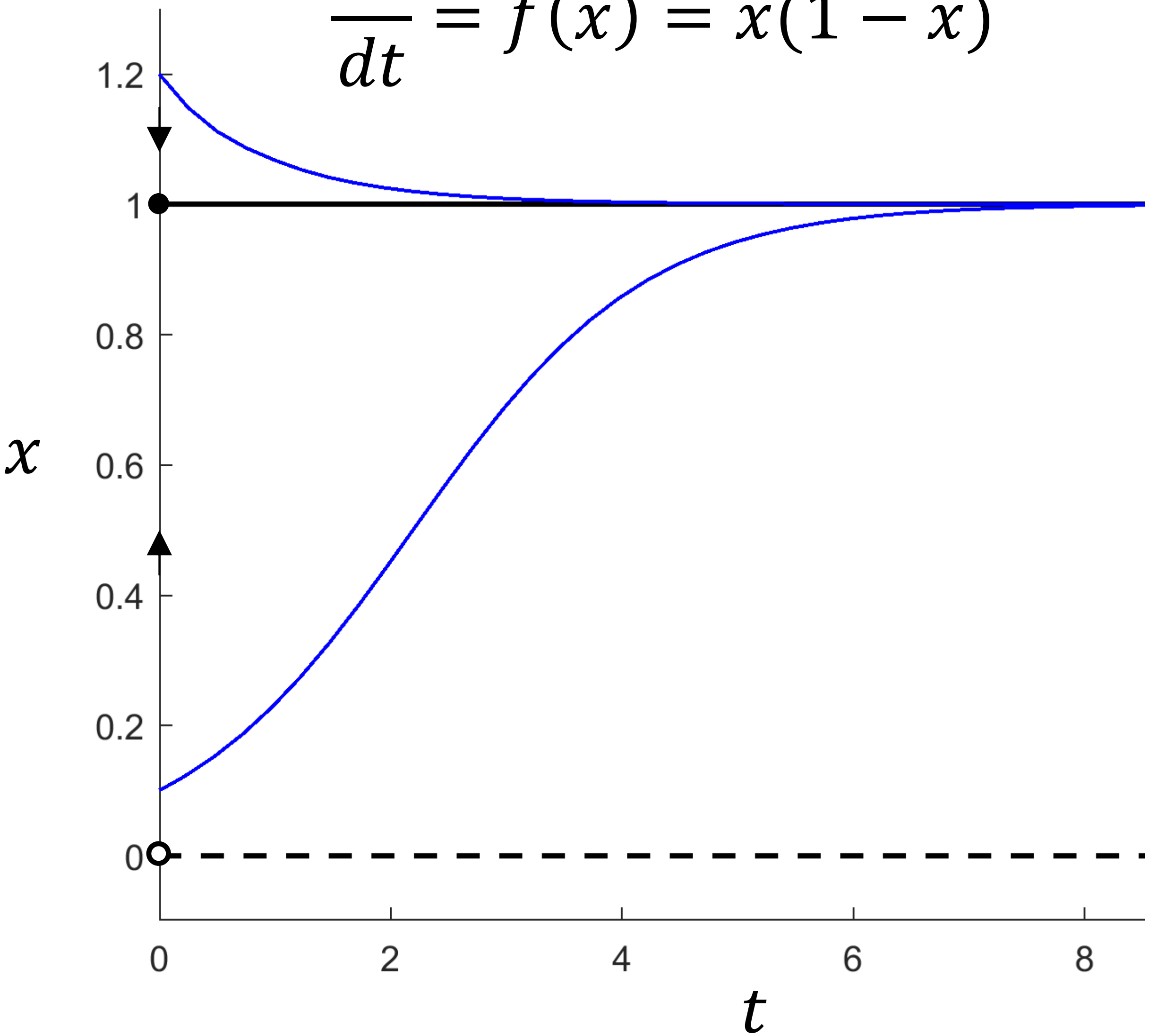


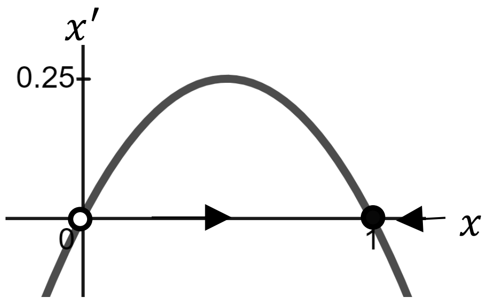

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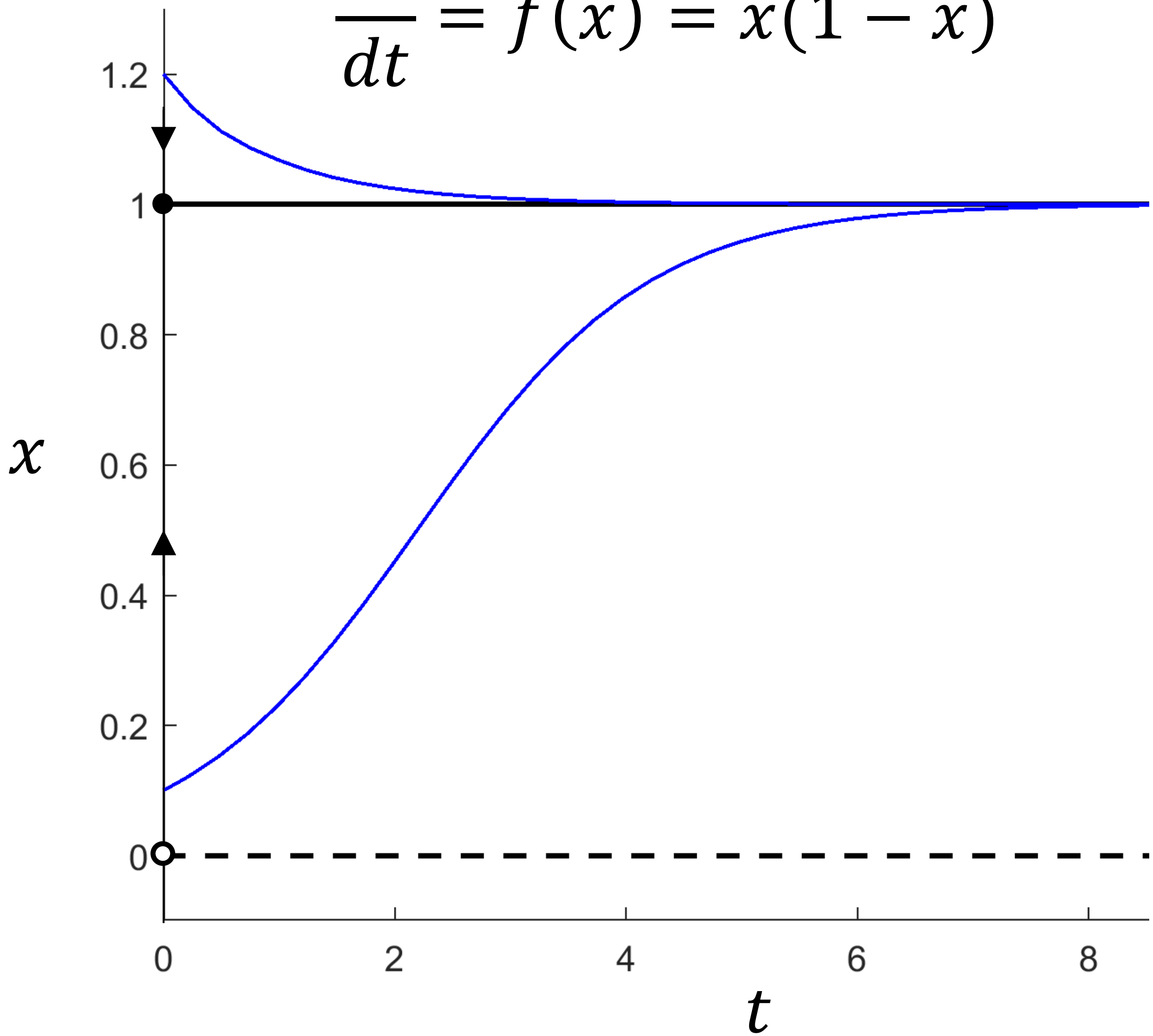


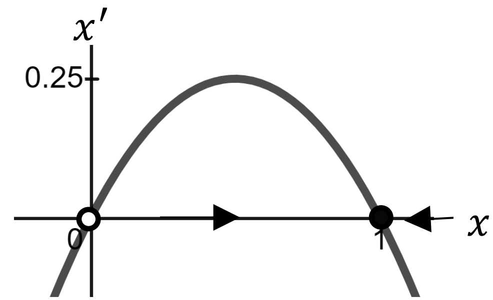


**Flow function**

$$\varphi : \mathbb{R} \times X \to X$$

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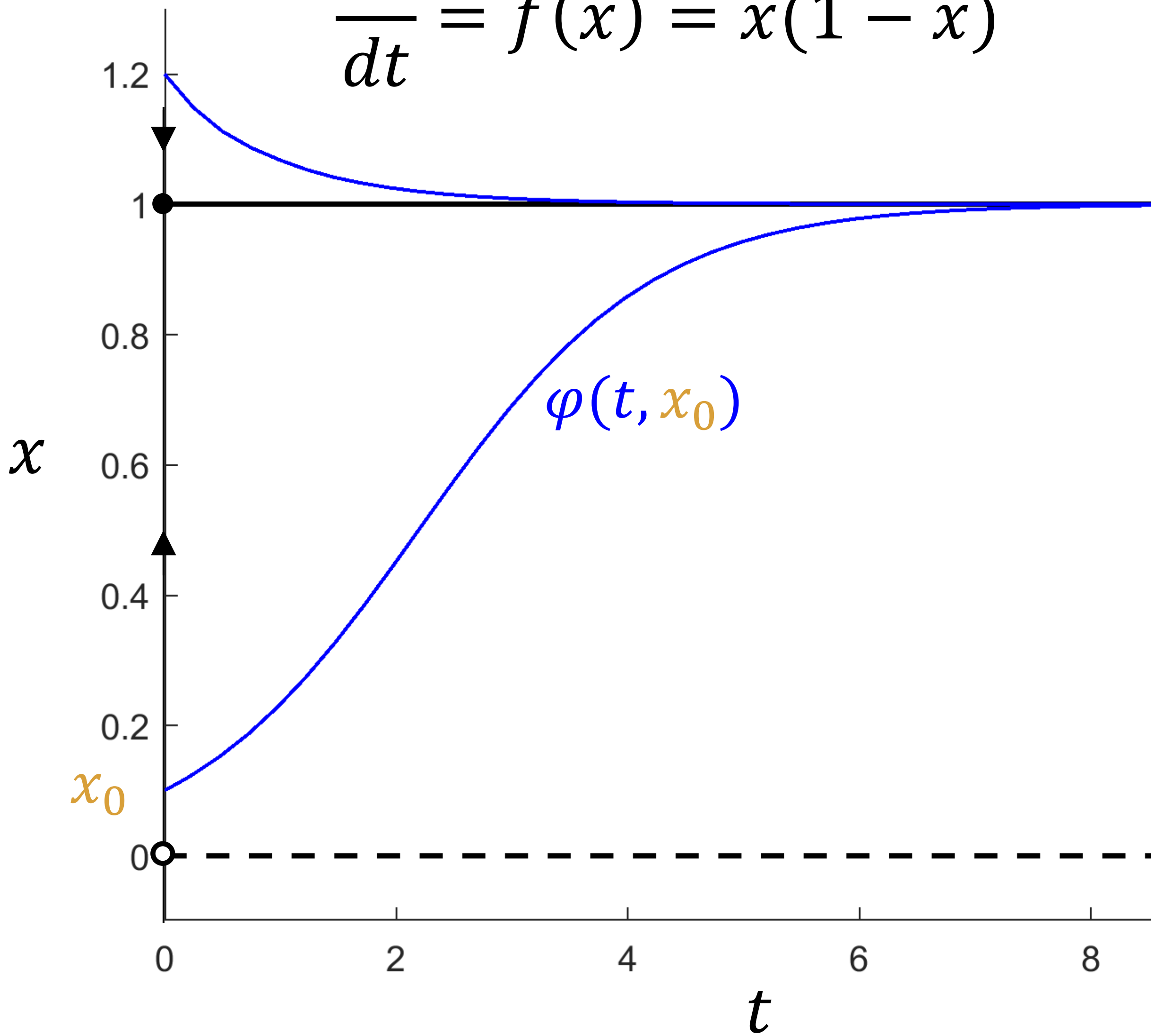


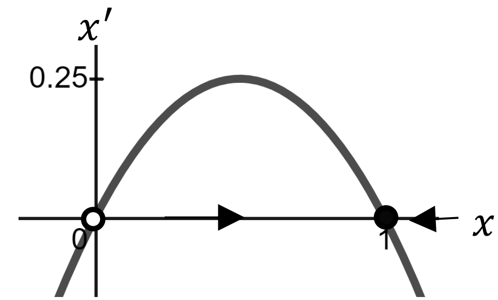


**Flow function**

$$\varphi: \mathbb{R} \times X \to X$$

$$\varphi(t, x_0)$$

solves IVP

$$\begin{cases} x' = f(x) \\ x(0) = x_0 \end{cases}$$

# Flow-kick harvest of logistic

**Flow-kick map**

$$\Phi_{\tau,\kappa}(x) = \varphi(\tau, x) + \kappa$$

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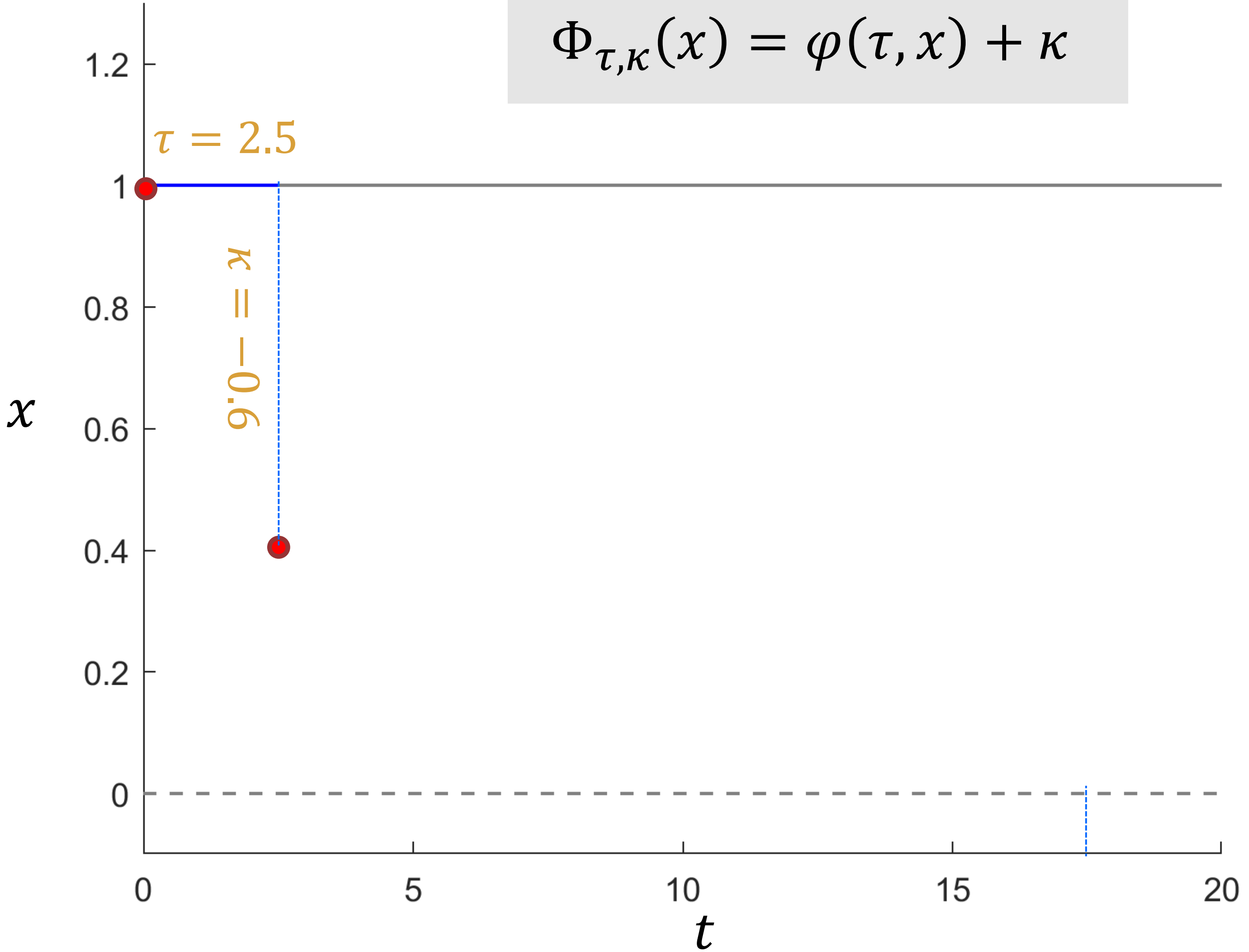

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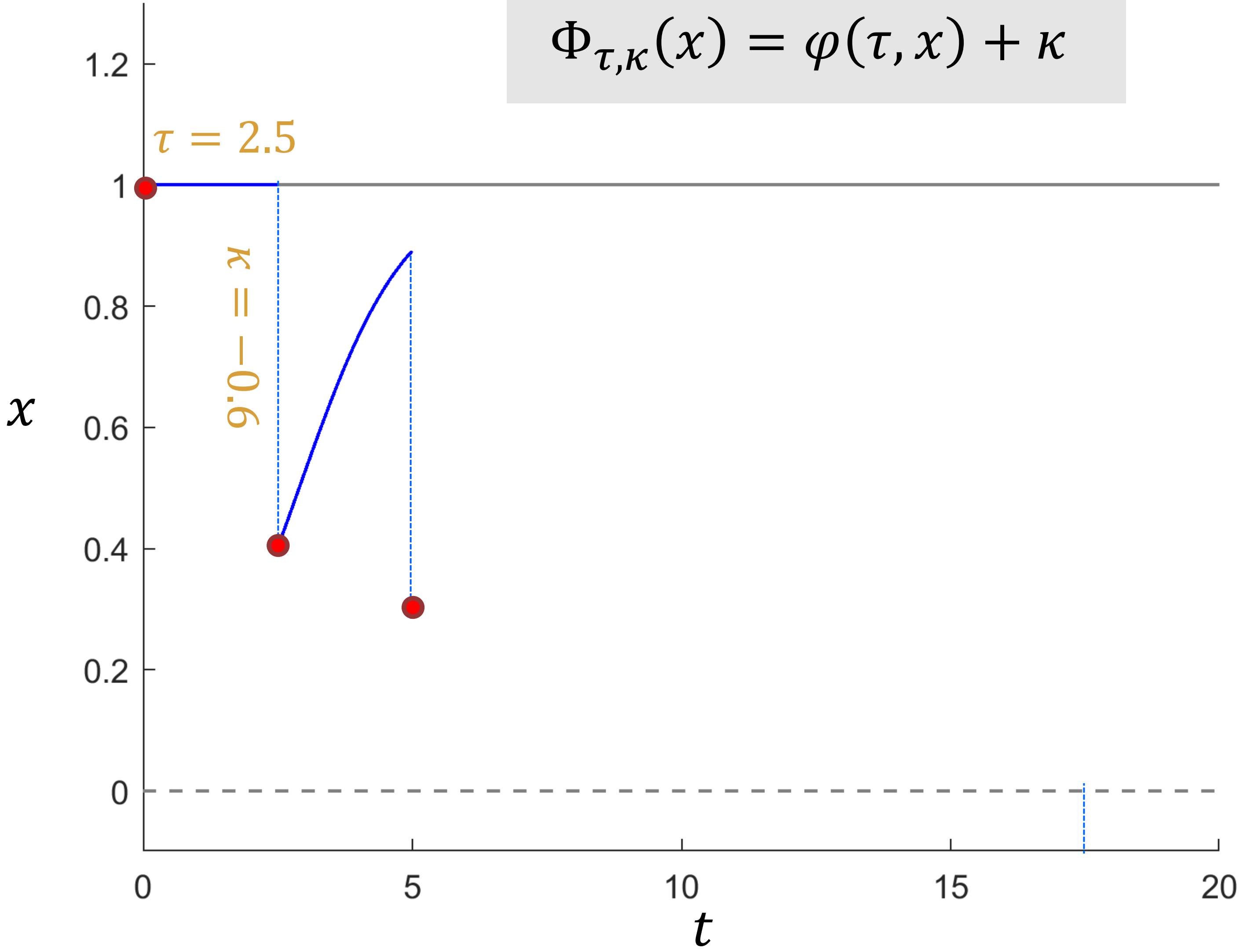

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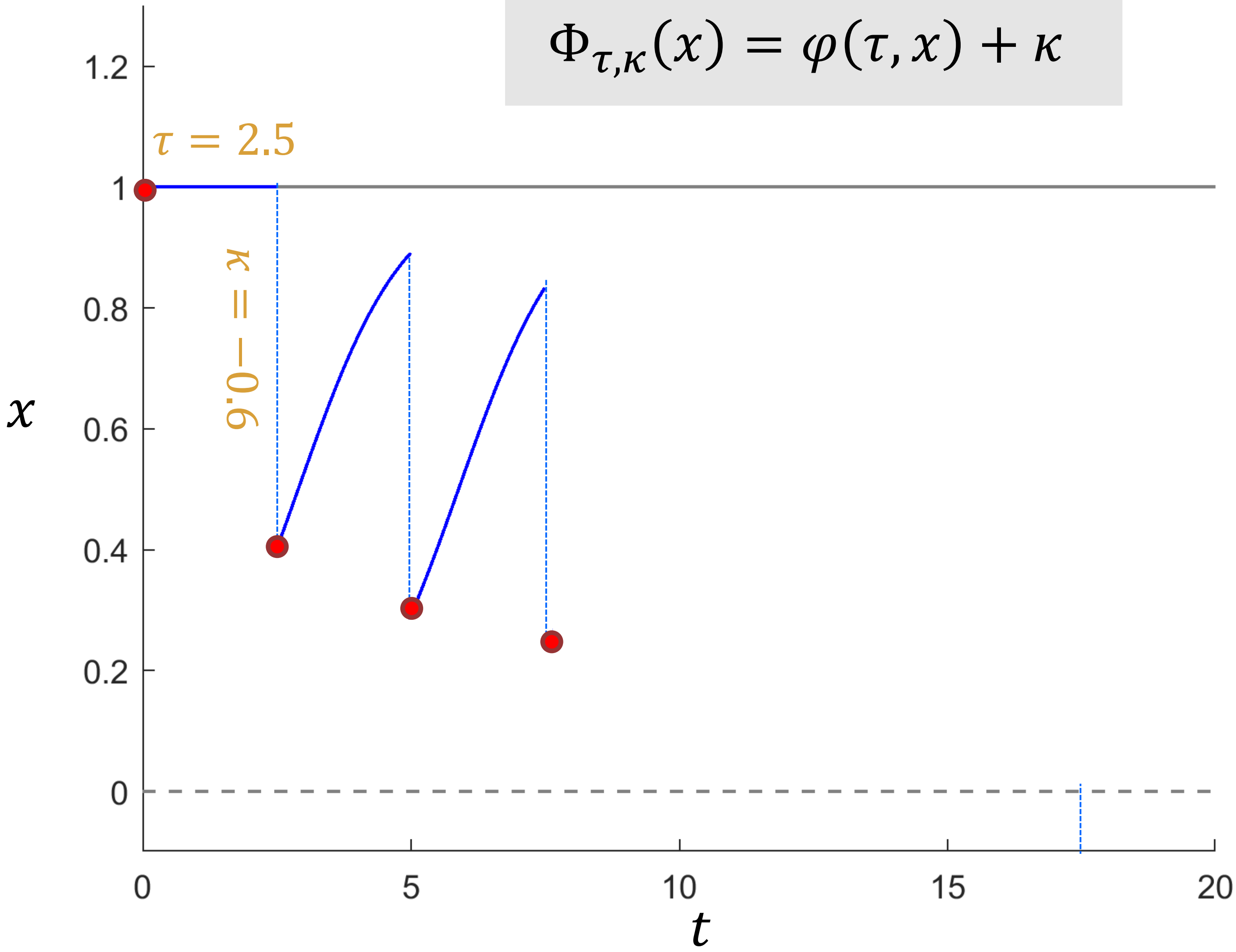

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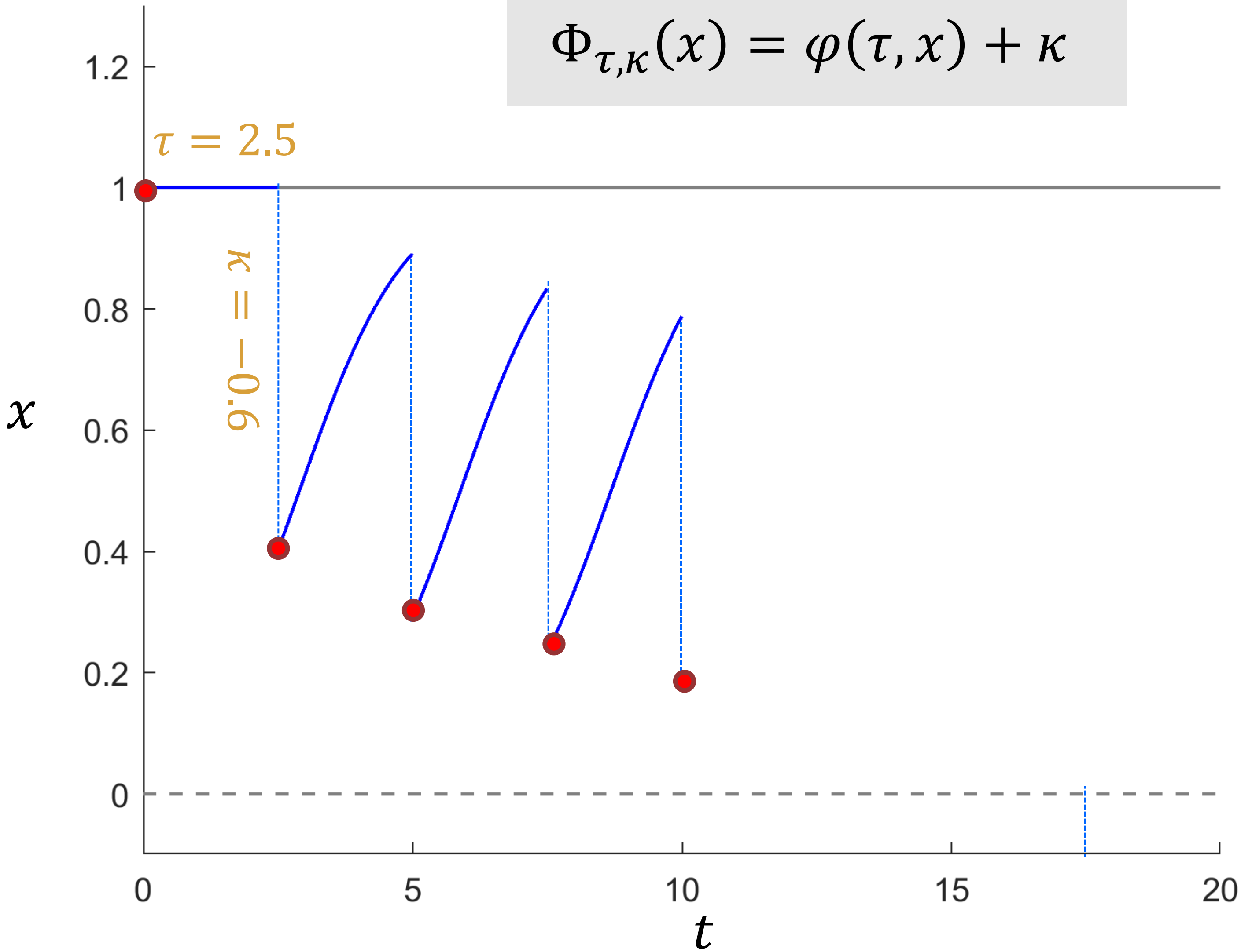

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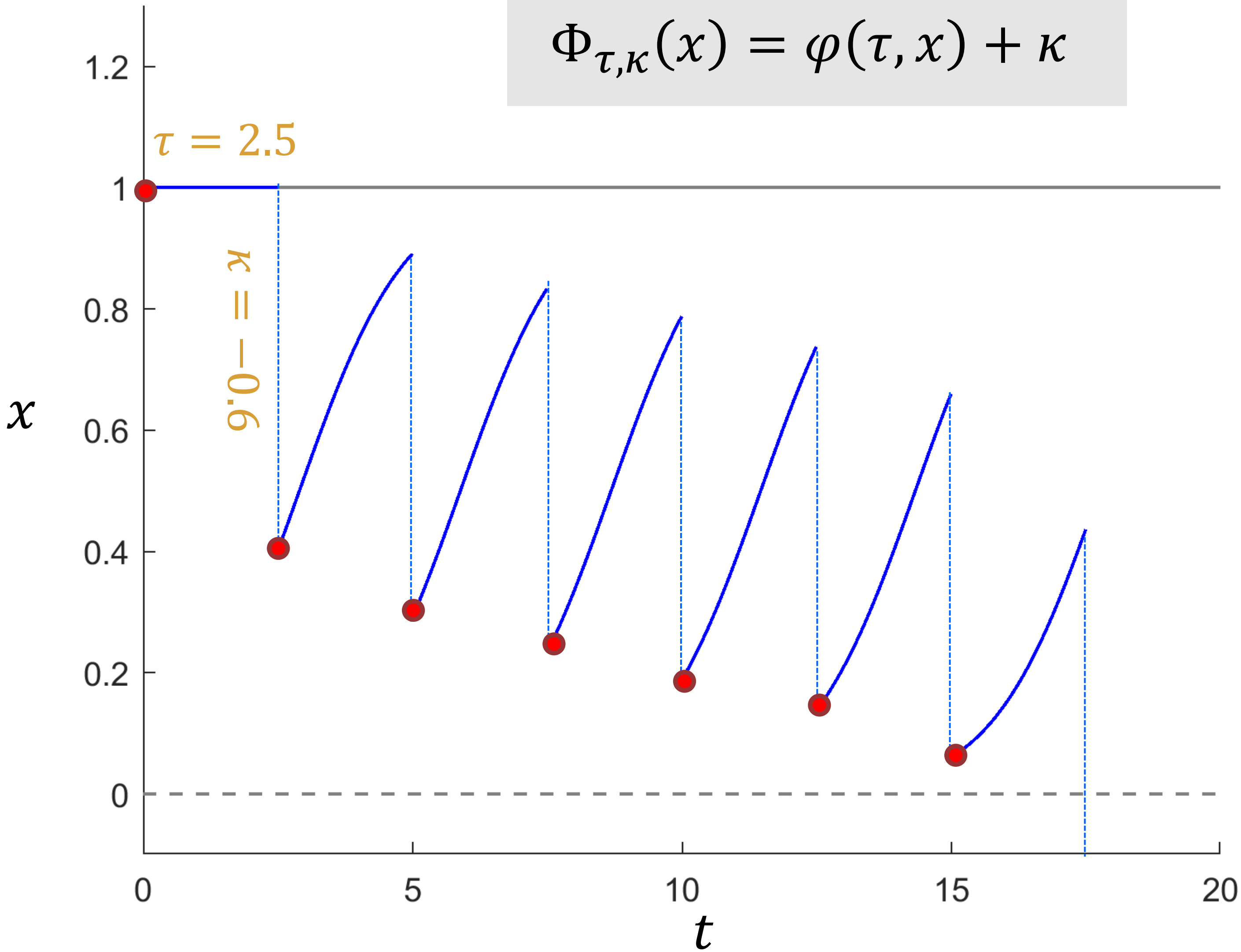

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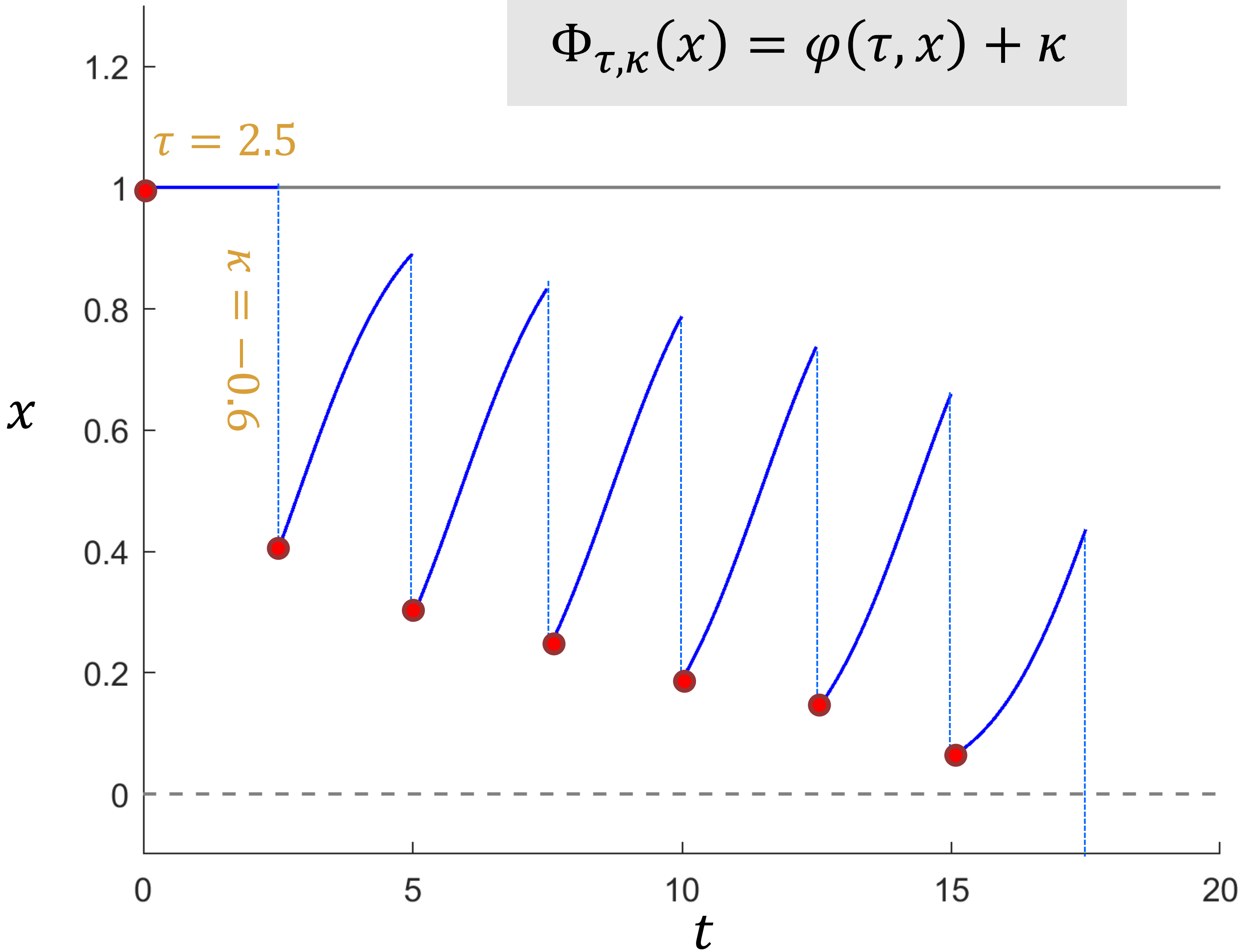


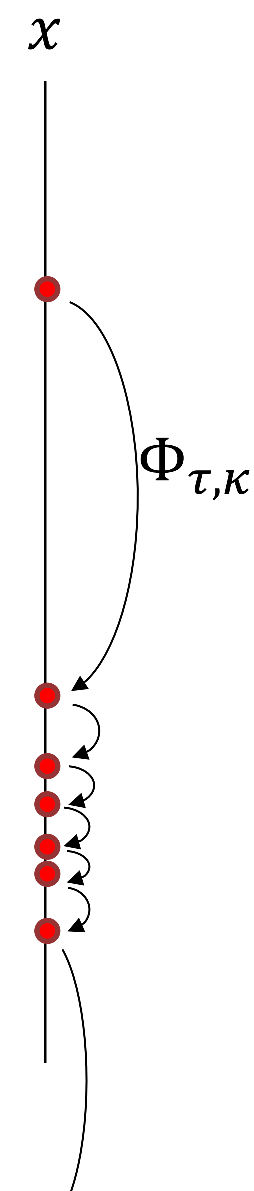

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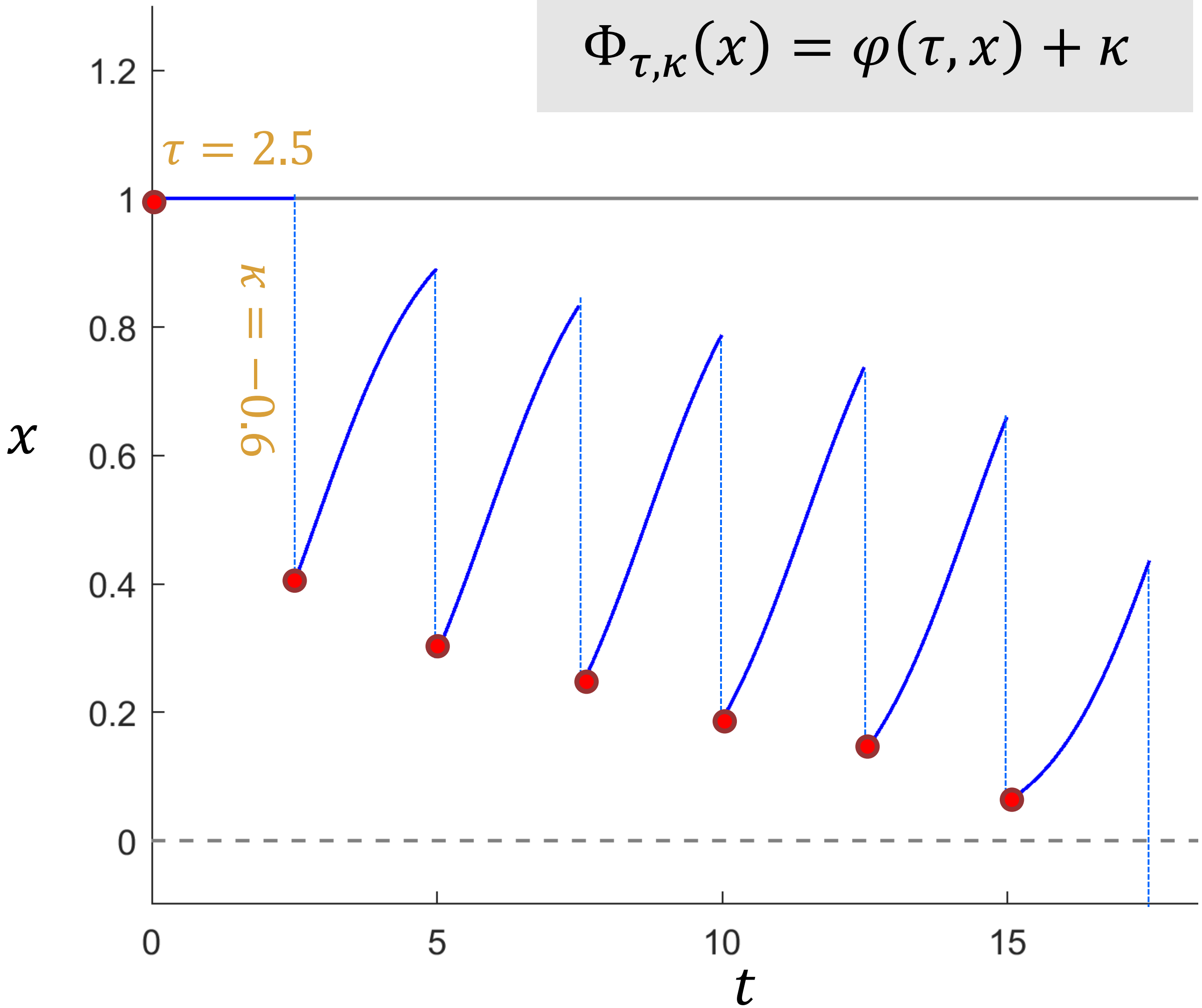
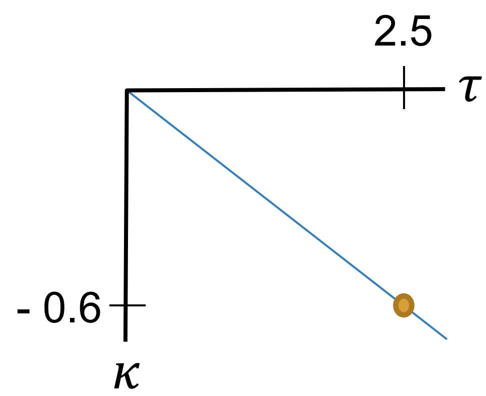



**Disturbance space**



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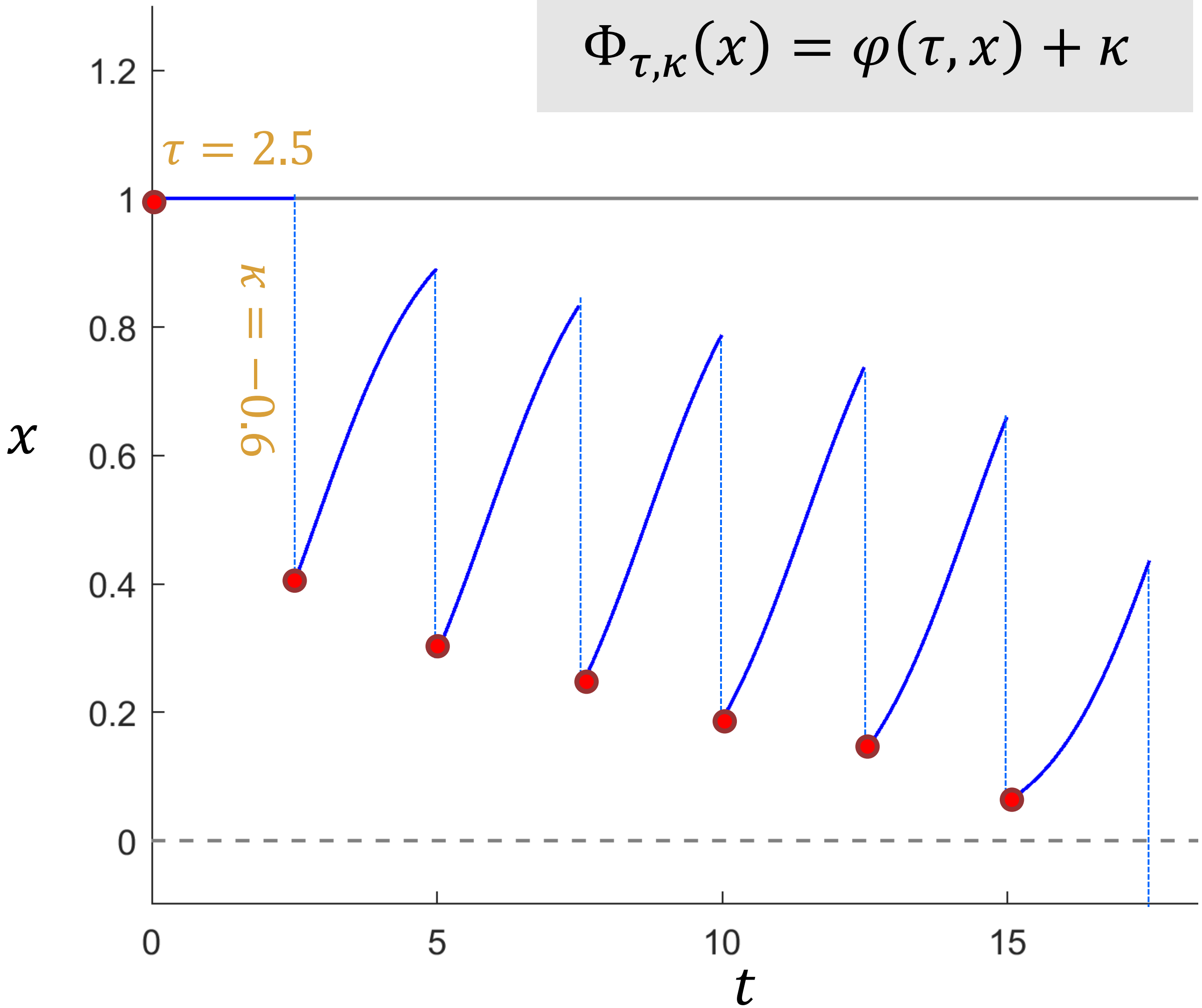
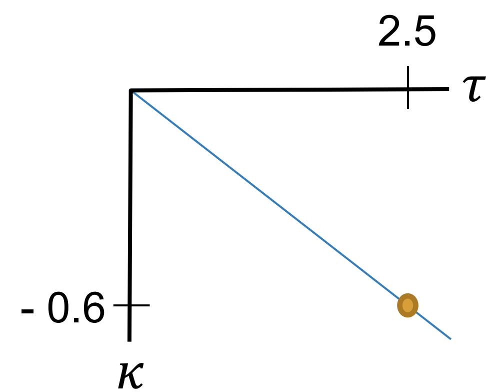



**Disturbance space**



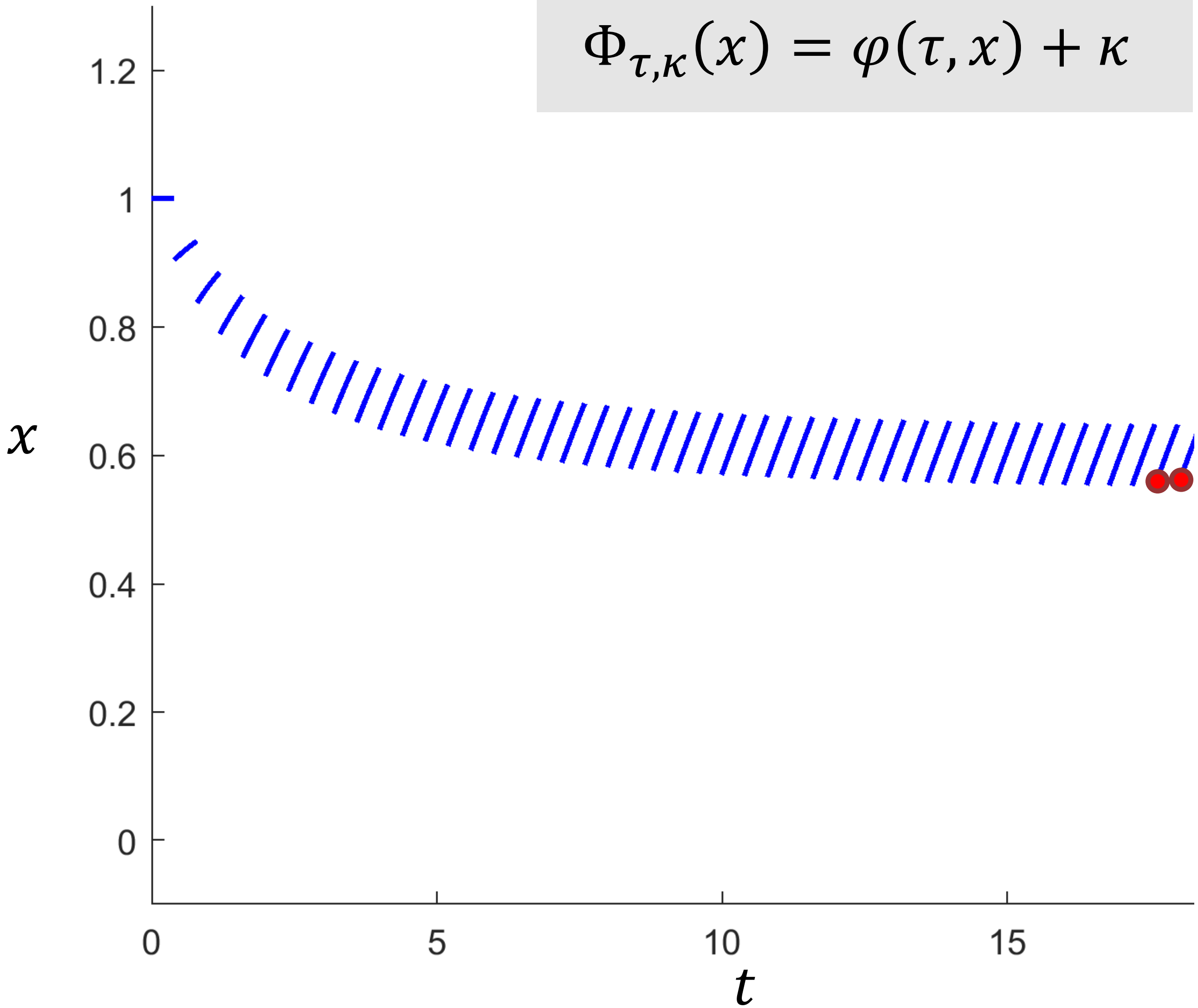

**Disturbance rate**

$$\frac{\kappa}{\tau} = -0.24$$

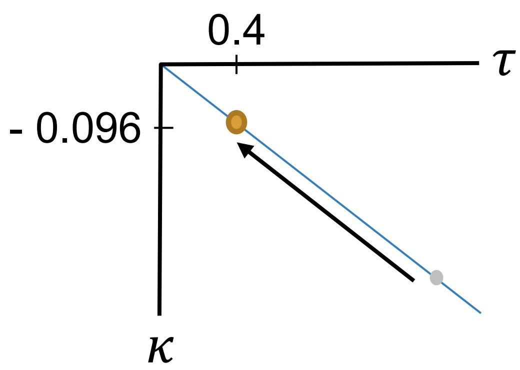
# Flow-kick harvest of logistic — more frequent

**Flow-kick map**

$$\Phi_{\tau,\kappa}(x) = \varphi(\tau, x) + \kappa$$



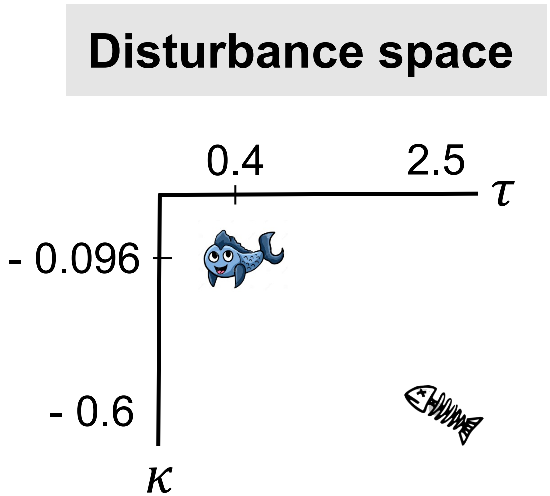

**Disturbance space**




**Disturbance rate**

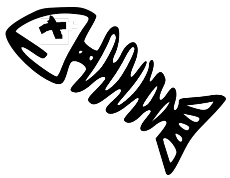
$$\frac{\kappa}{\tau} = -0.24$$

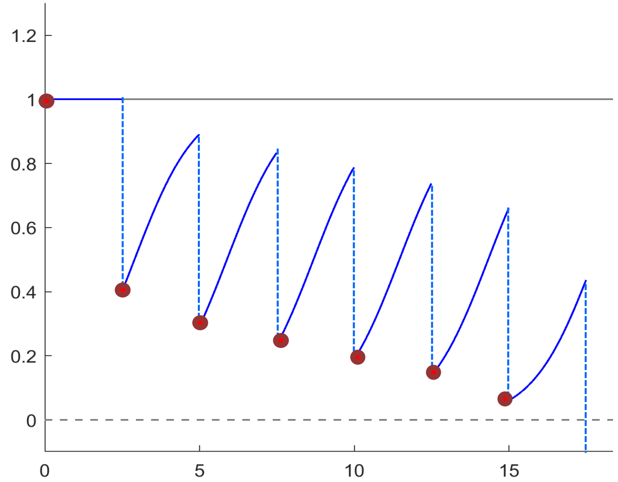
# Which harvest frequencies, sizes are sustainable?



# Which harvest frequencies, sizes are sustainable?

| | A | B | C |
|---|---|---|---|
| 1 | | | |
| 2 | | Flow time: | Kick: |
| 3 | | 2.5 | -0.6 |
| 4 | | | |
| 5 | **Pop. Size (x)** | **x after flow** | **x after kick** |
| 6 | 1.000 | 1.000 | 0.400 |
| 7 | 0.400 | 0.890 | 0.290 |
| 8 | 0.290 | 0.833 | 0.233 |
| 9 | 0.233 | 0.787 | 0.187 |
| 10 | 0.187 | 0.737 | 0.137 |
| 11 | 0.137 | 0.660 | 0.060 |
| 12 | 0.060 | 0.436 | **-0.164** |
| 13 | **-0.164** | 2.391 | 1.791 |
| 14 | 1.791 | 1.038 | 0.438 |



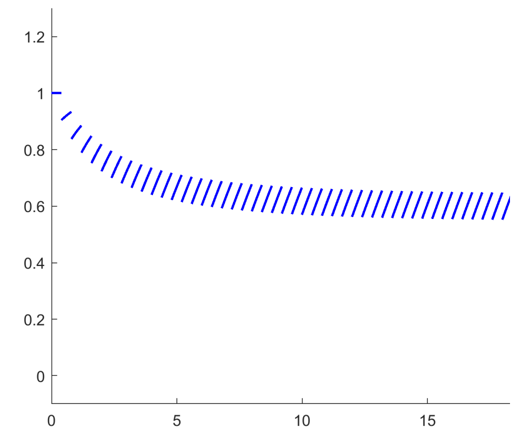



fx =A7/(A7+(1-A7)*EXP(-$B$3))

solution to logistic ODE

# Which harvest frequencies, sizes are sustainable?

| | A | B | C |
|---|---|---|---|
| 1 | | | |
| 2 | | Flow time: | Kick: |
| 3 | | 0.4 | -0.096 |
| 4 | | | |
| 5 | **Pop. Size (x)** | **x after flow** | **x after kick** |
| 6 | 1.000 | 1.000 | 0.904 |
| 7 | 0.904 | 0.934 | 0.838 |
| 8 | 0.838 | 0.885 | 0.789 |
| 9 | 0.789 | 0.848 | 0.752 |
| 10 | 0.752 | 0.819 | 0.723 |
| 11 | 0.723 | 0.796 | 0.700 |
| 12 | 0.700 | 0.776 | 0.680 |
| 13 | 0.680 | 0.761 | 0.665 |
| 14 | 0.665 | 0.747 | 0.651 |
| 15 | 0.651 | 0.736 | 0.640 |
| 16 | 0.640 | 0.726 | 0.630 |
| 17 | 0.630 | 0.718 | 0.622 |



# Which harvest frequencies, sizes are sustainable?

https://tinyurl.com/fishkicks

# An analytic solution

**Flow-kick map**

$$\Phi_{\tau,\kappa}(x) = \varphi(\tau, x) + \kappa$$

Fixed point condition: $\Phi_{\tau,\kappa}(x) = x$

$$\varphi(\tau, x) + \kappa = x$$

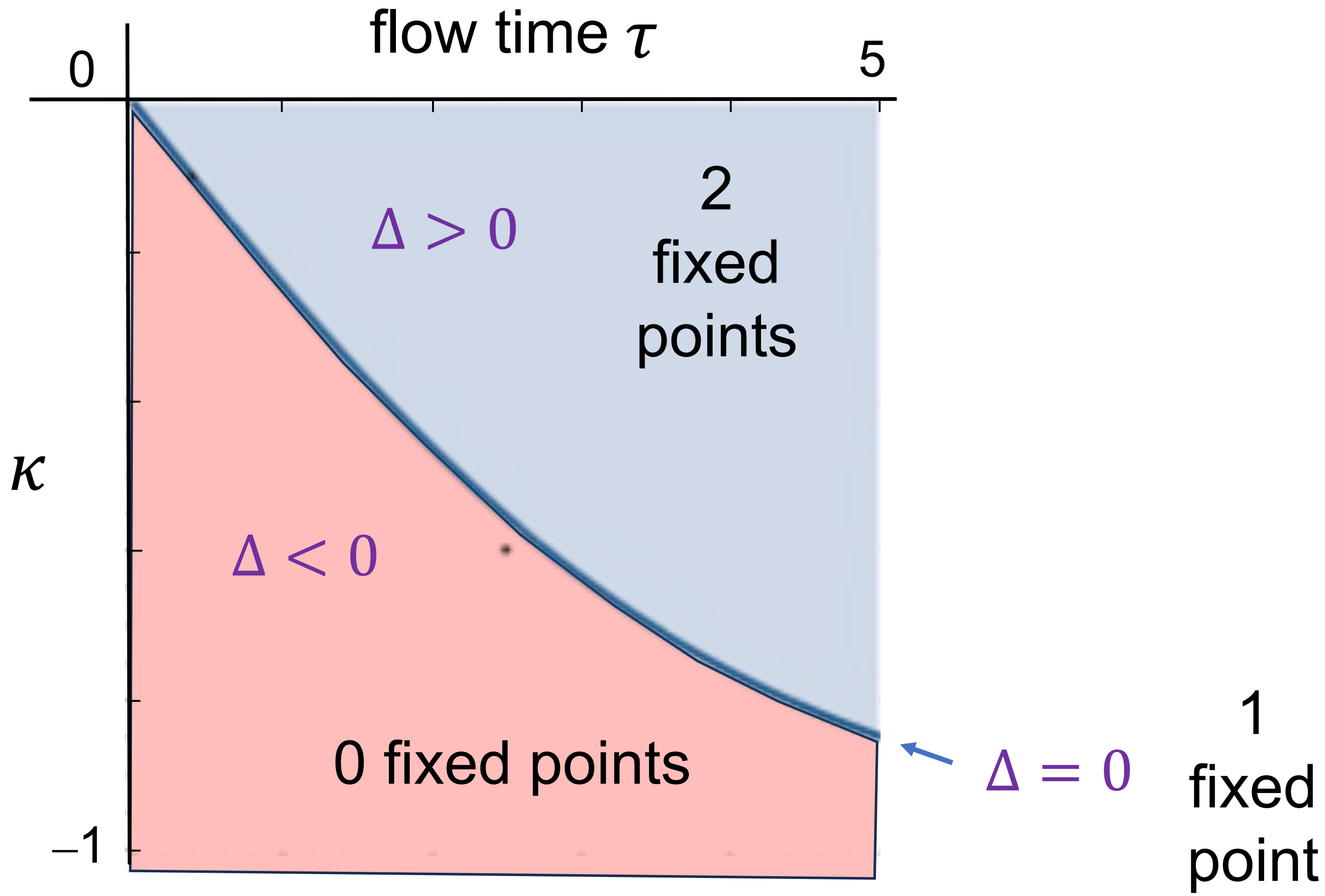
$$\frac{x}{x + (1-x)e^{-\tau}} + \kappa = x$$

$$\vdots$$

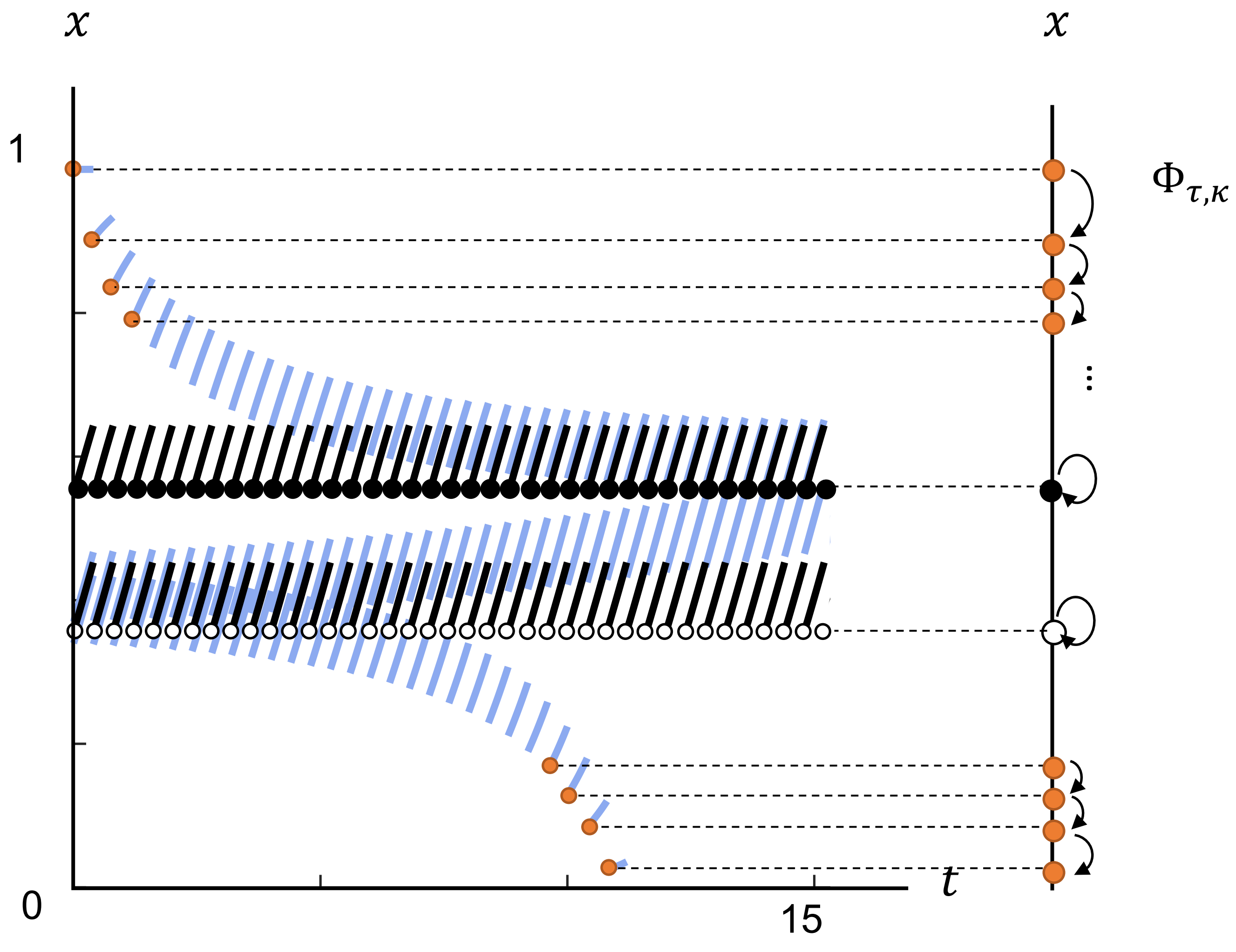
$$x^2 + (\kappa + 1)x + \frac{\kappa e^{-\tau}}{1 - e^{-\tau}} = 0$$

$$x = \frac{1}{2}\left[-\kappa - 1 \pm \sqrt{(\kappa+1)^2 - \frac{4\kappa e^{-\tau}}{1 - e^{-\tau}}}\right] \quad \Delta$$

# An analytic solution

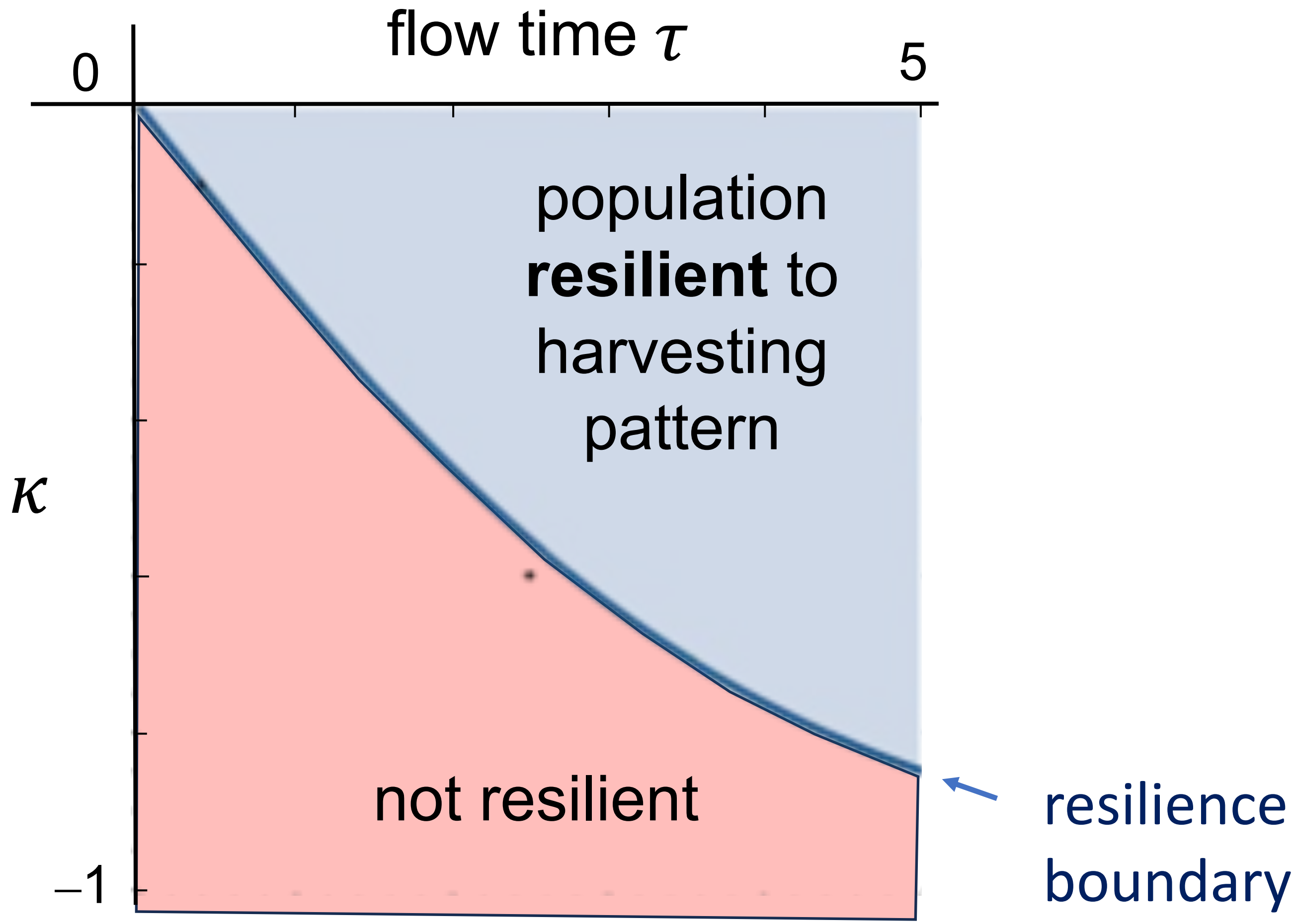


# Two fixed points



# Resilience metrics

Meyer et al. 2018, *Nat. Sustain.* 1



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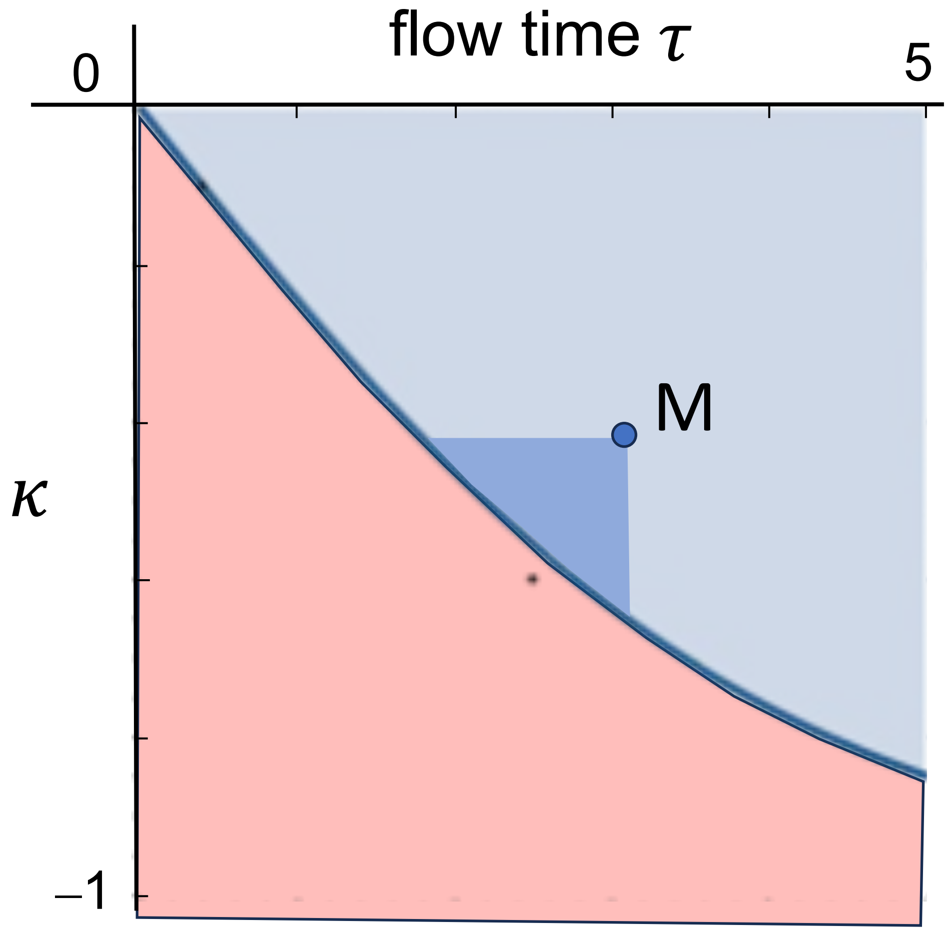



resilience to changes in management strategy

Part 1: Resilience frameworks

Part 2: Flow-kick models for quantifying resilience to repeated disturbances

Part 3: Climate applications

# Resilience of the AMOC

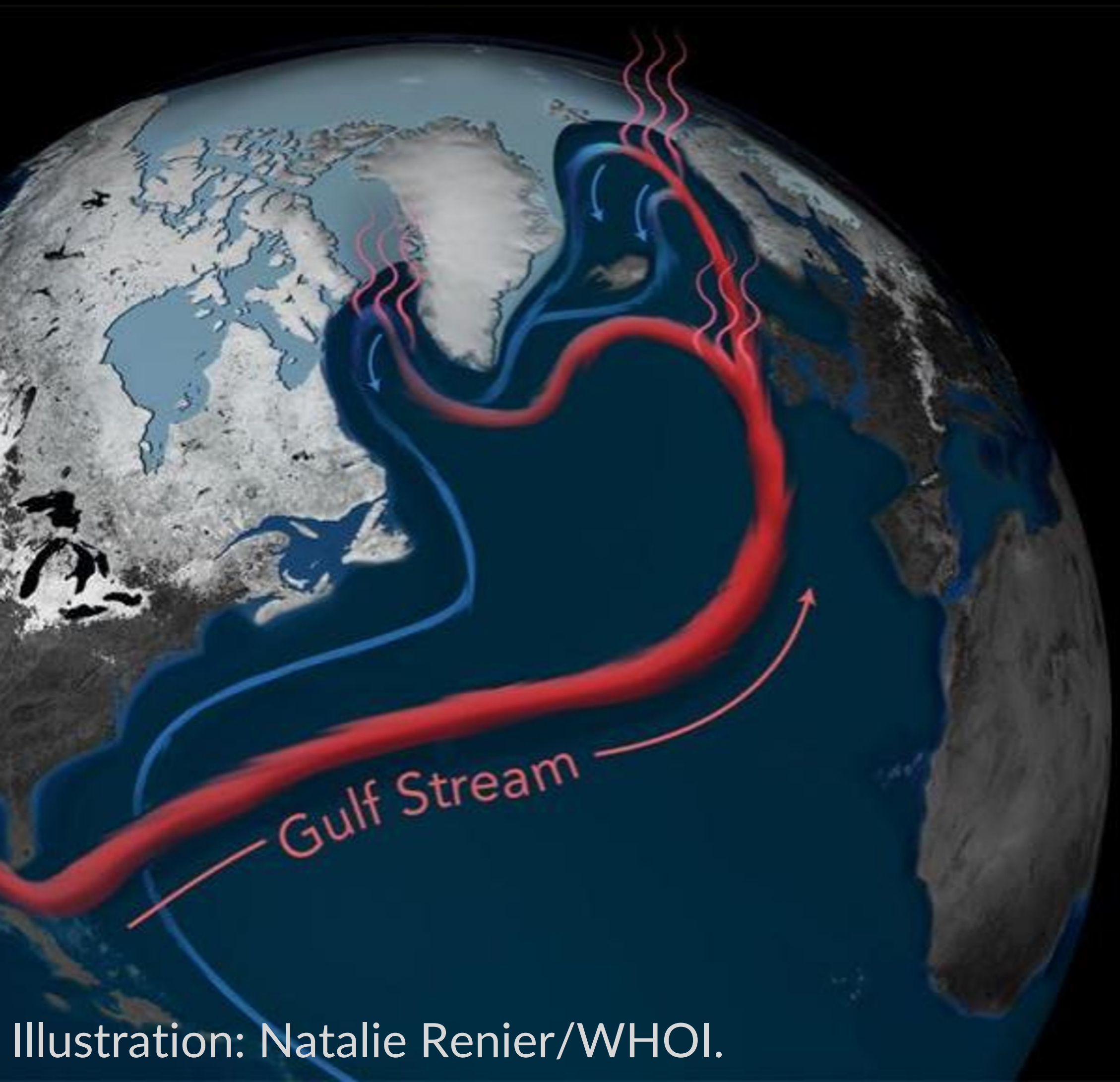



Illustration: Natalie Renier/WHOI.

# Resilience of the AMOC

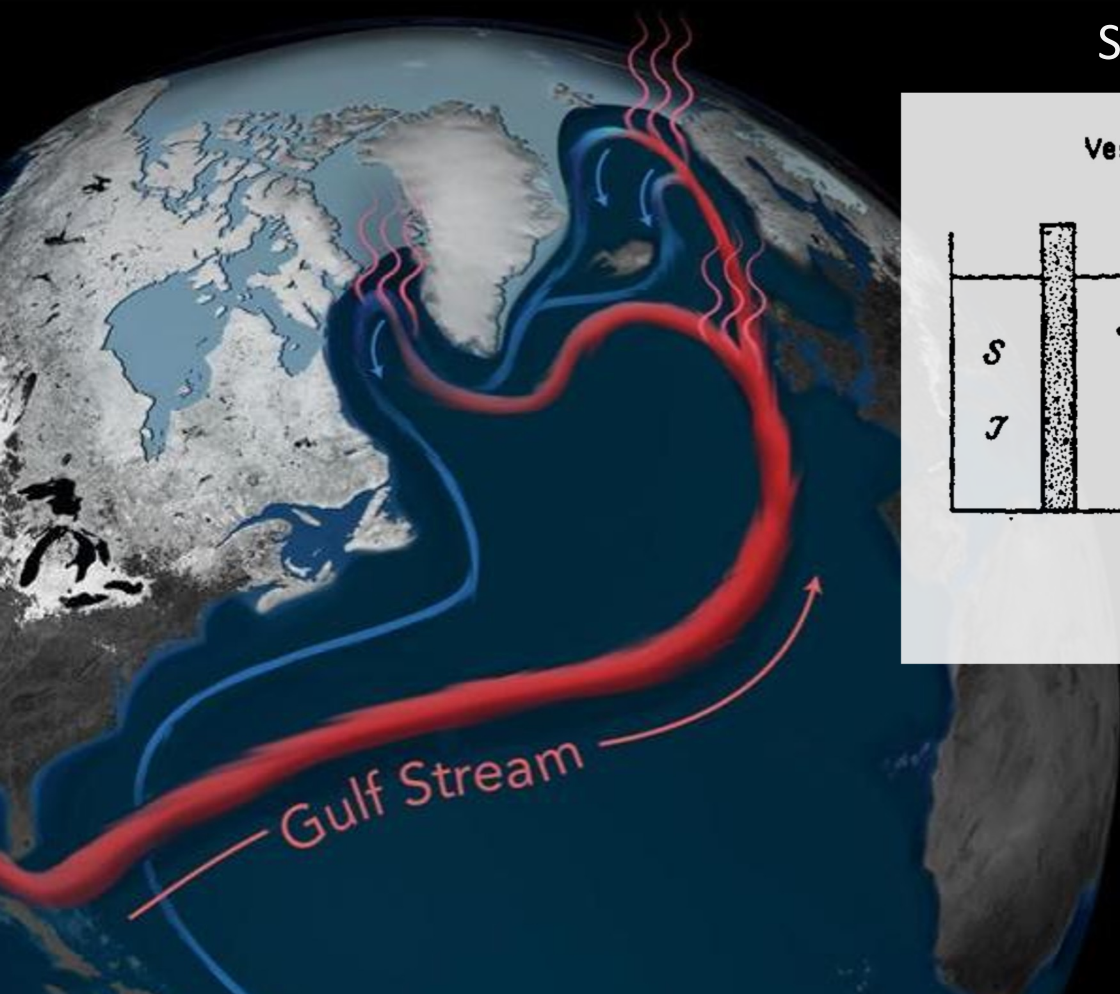
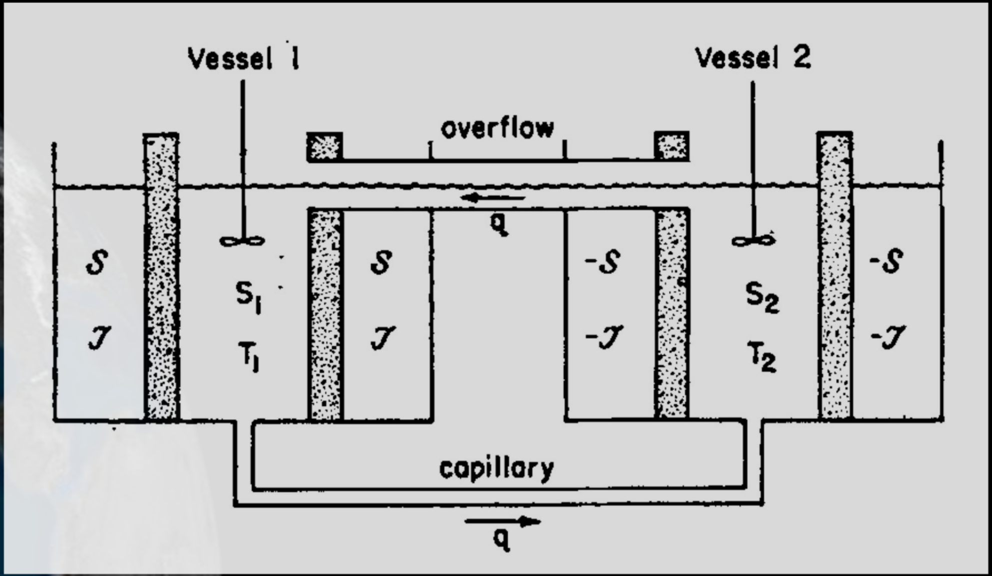
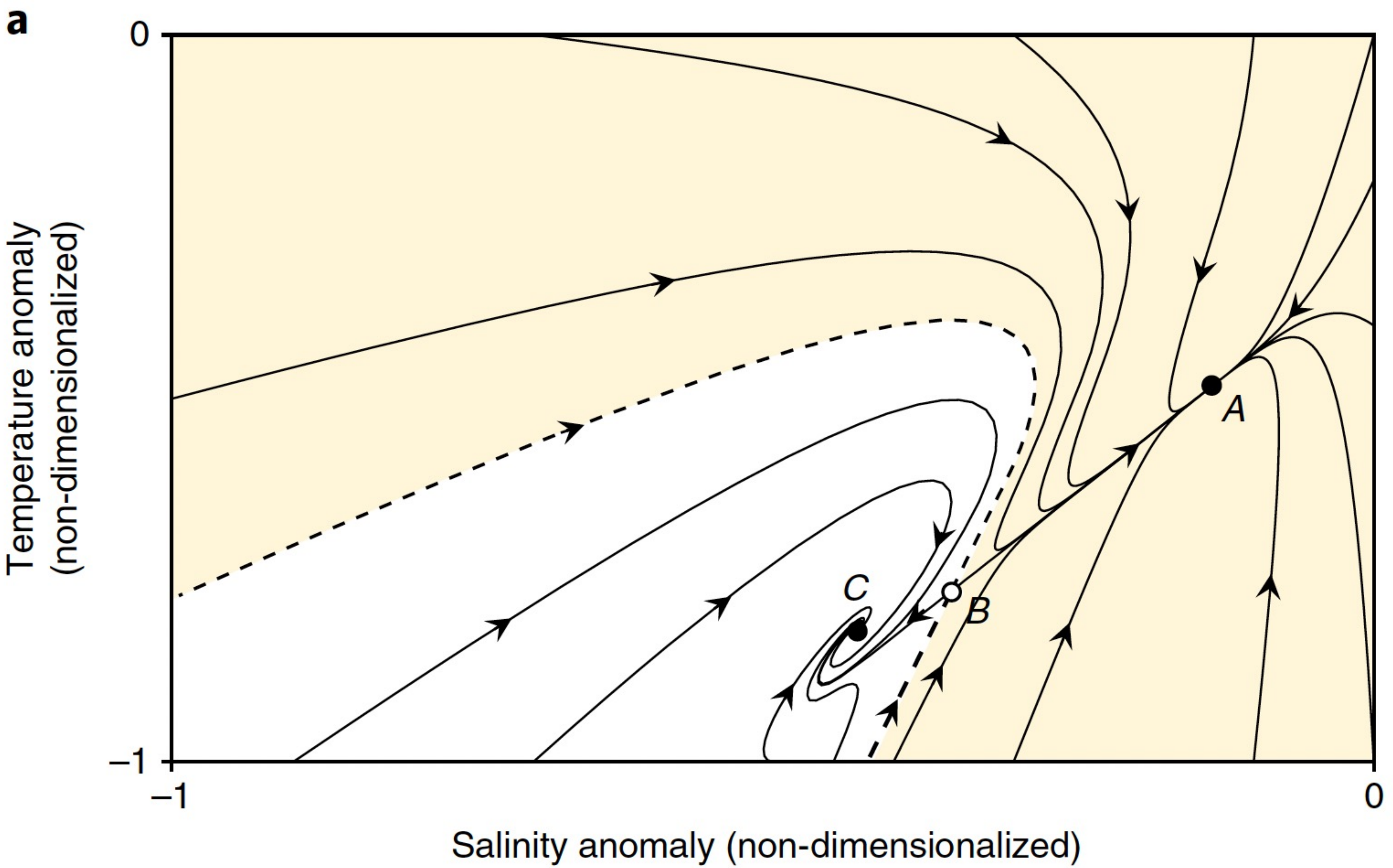



Illustration: Natalie Renier/WHOI.

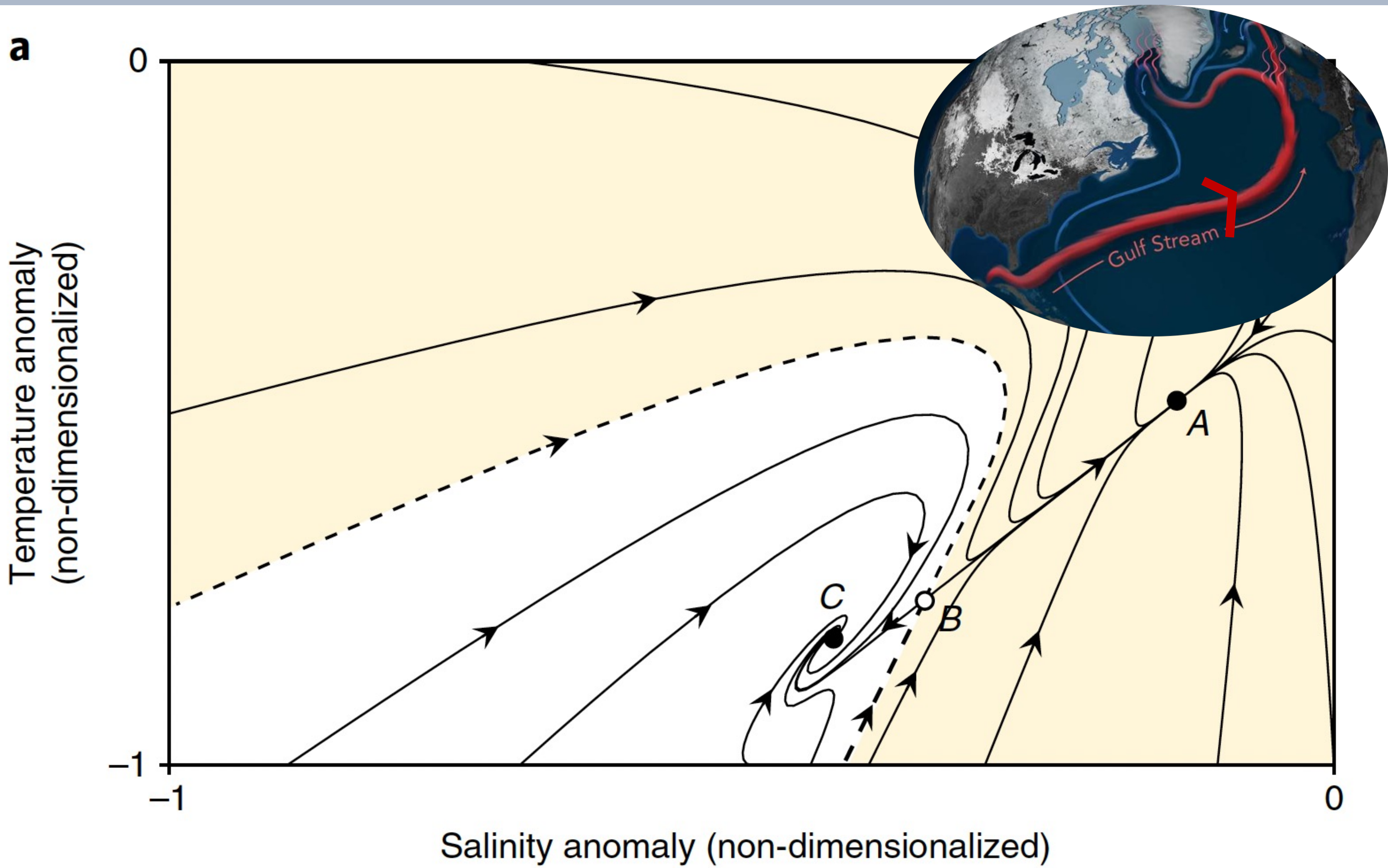
Stommel 1961, *Tellus* 13(2)



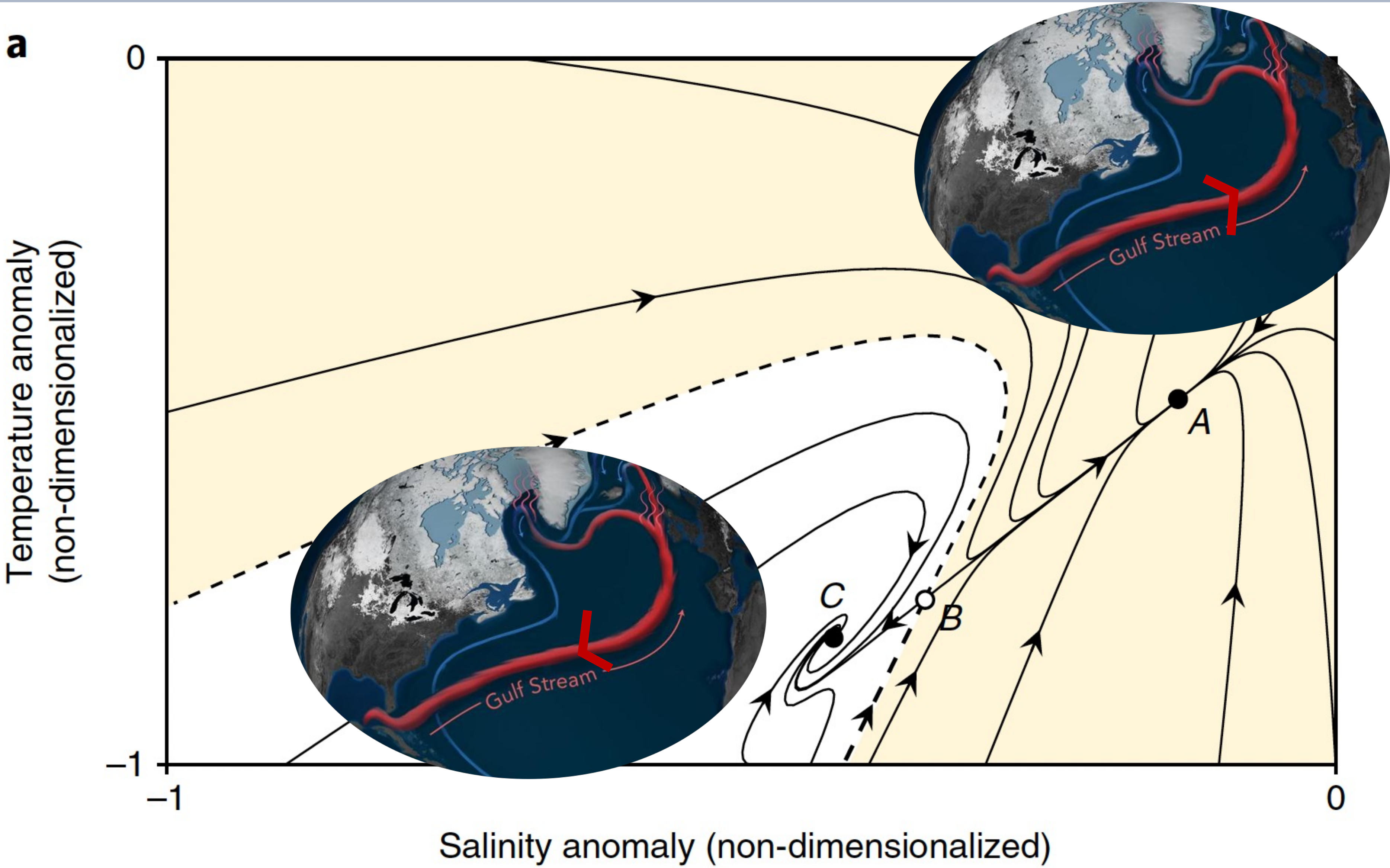
# Undisturbed flow dynamics



# Undisturbed flow dynamics



# Undisturbed flow dynamics



# Salinity disturbances

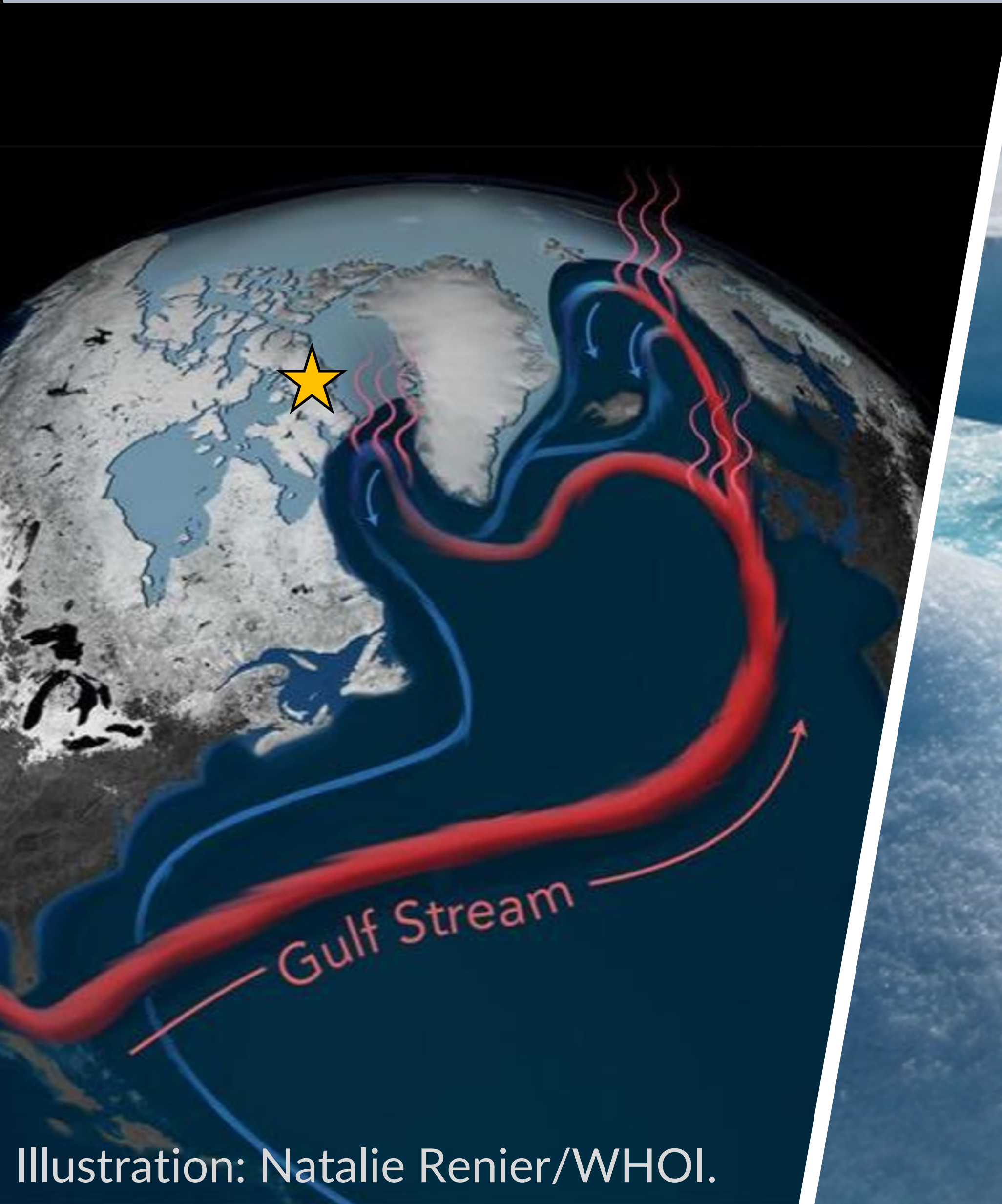
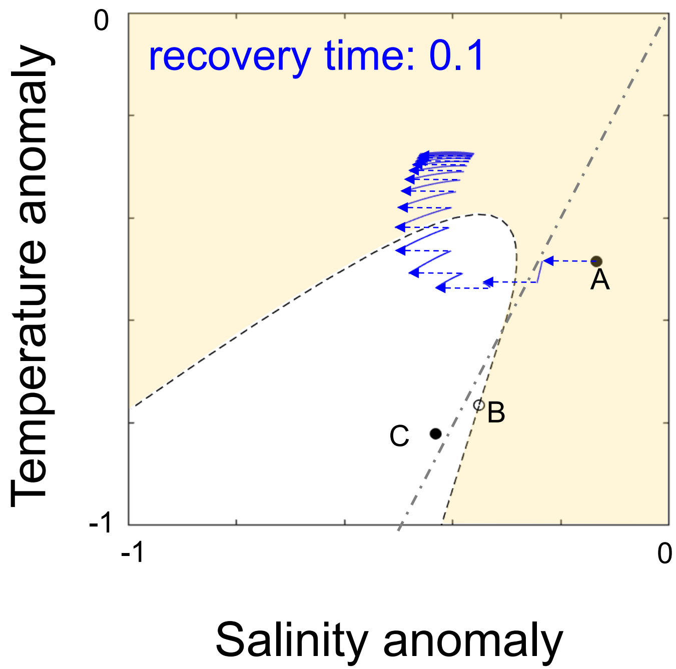


Illustration: Natalie Renier/WHOI.


Gifford Miller / CU Boulder

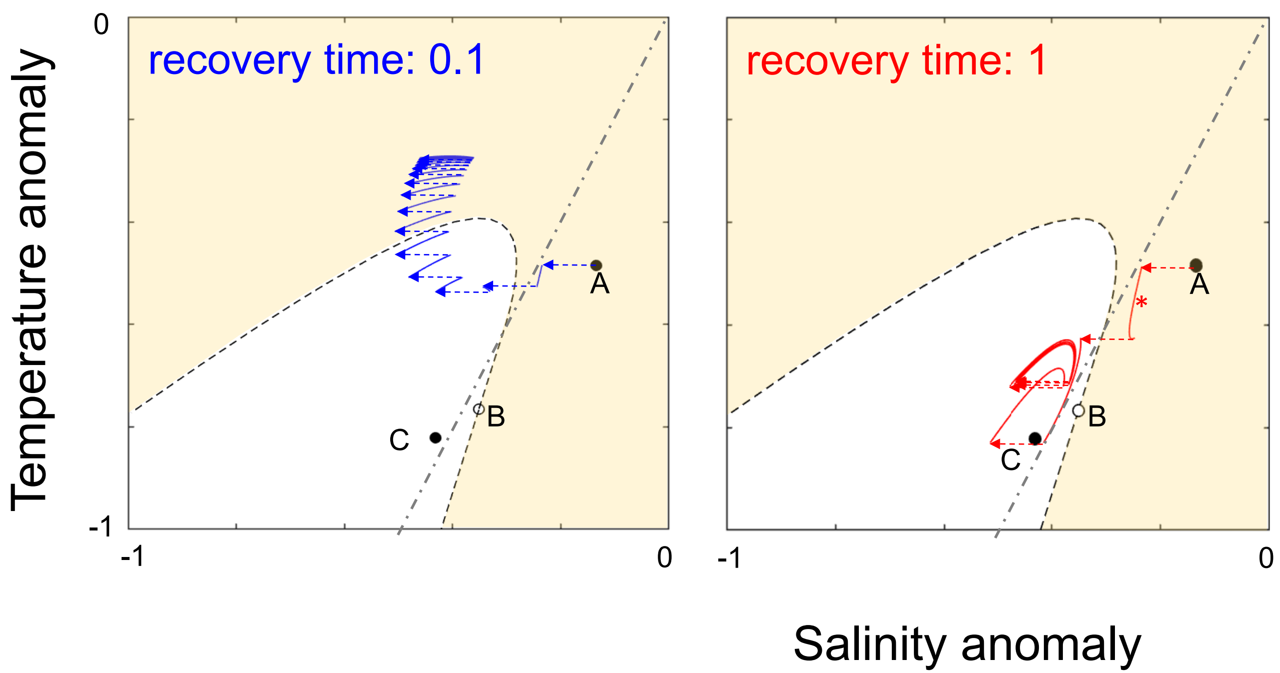
# Salinity disturbances

Meyer et al. 2018, *Nat. Sustain.* 1



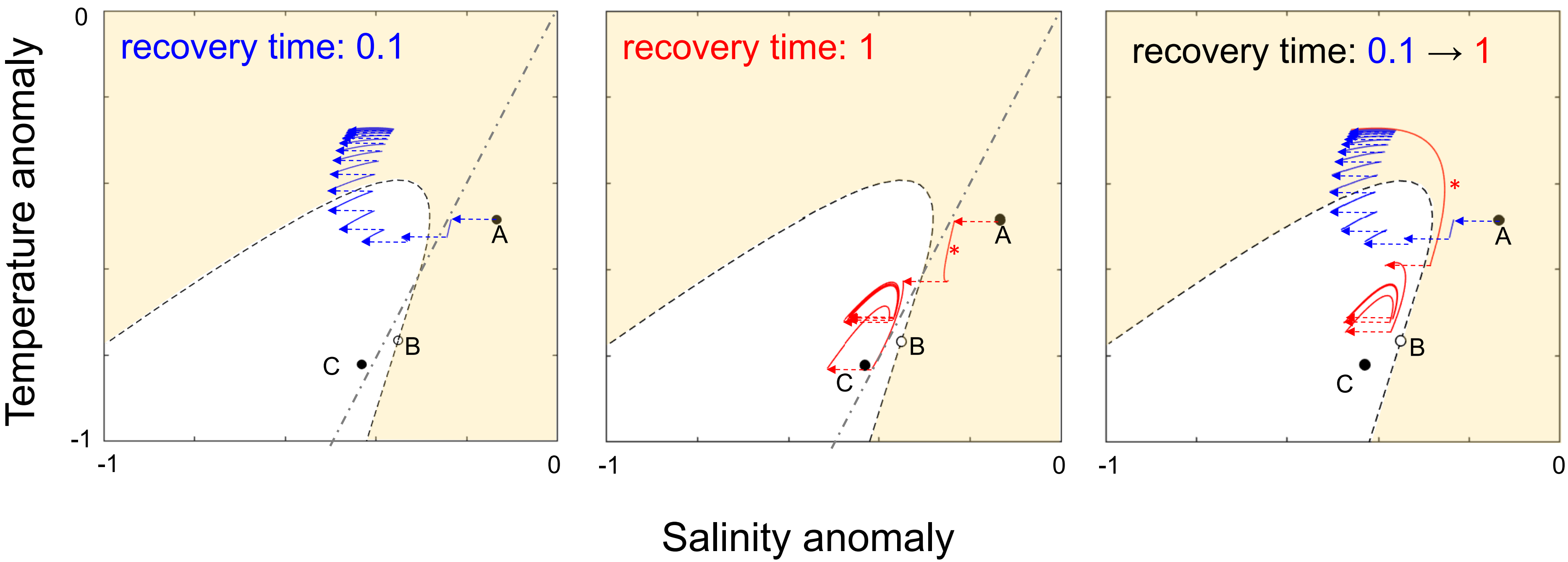
# Salinity disturbances

Meyer et al. 2018, *Nat. Sustain.* 1



# Salinity disturbances

Meyer et al. 2018, *Nat. Sustain.* 1



# Beyond basins…

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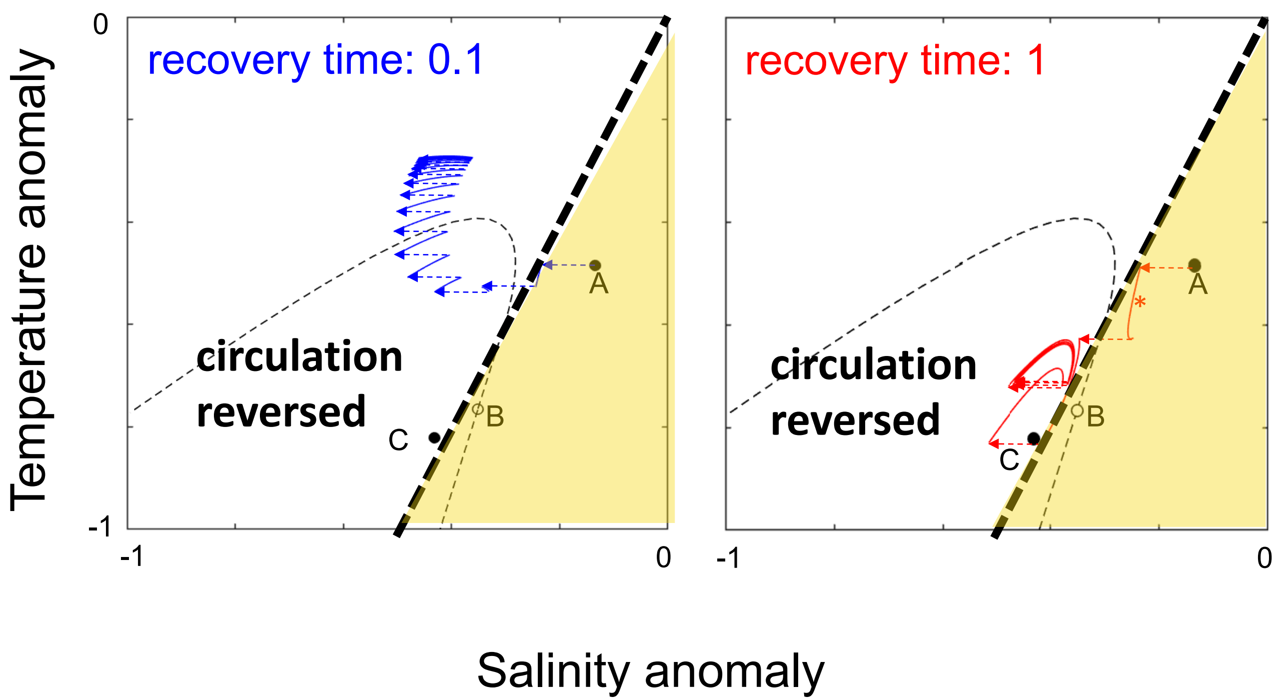
WILEY Natural Resource Modeling

# Resilience of socially valued properties of natural systems to repeated disturbance: A framework to support value-laden management decisions

Mary Lou Zeeman[1] | Katherine Meyer[2] | Erika Bussmann[3] | Alanna Hoyer-Leitzel[4] | Sarah Iams[5] | Ian J. Klasky[6] | Victoria Lee[7] | Stephen Ligtenberg[7]

# Salinity disturbances

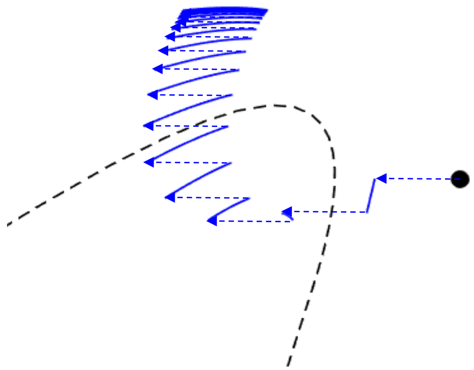
Meyer et al. 2018, *Nat. Sustain.* 1



# Computational Approaches

for finding flow-kick fixed points in $\mathbb{R}^n$

- Simulate flow-kick trajectories
  $\rightarrow$ discover (some) stable
  flow-kick fixed points

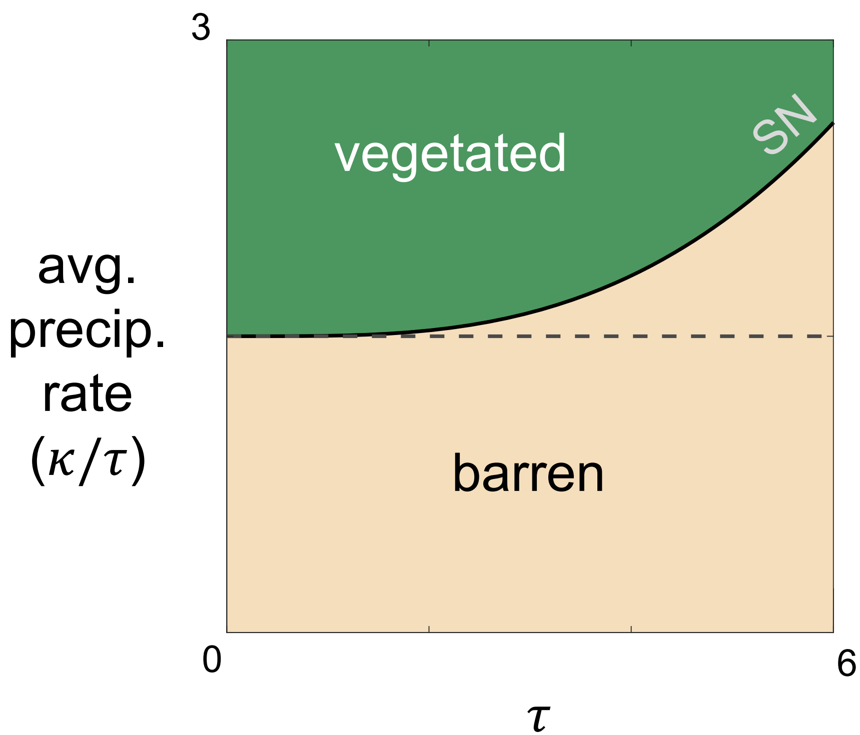


- Newton's method on fixed point condition $\Phi_{\tau,\kappa}(x) = x$
  (requires numerical approximation of $D\Phi_{\tau,\kappa}(x)$ )

# Example: Nonspatial Klausmeier Model

biomass $b' = wb^2 - mb$

water $w' = -w - wb^2$ + rain
(continuous $r$
or kicks $r\tau$)

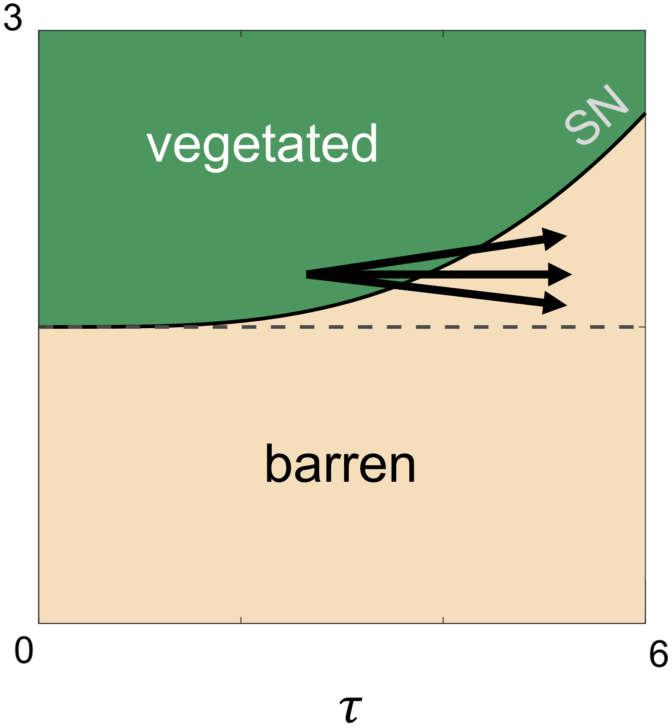


# Example: Nonspatial Klausmeier Model

biomass $b' = wb^2 - mb$

water $w' = -w - wb^2$ + rain
(continuous $r$
or kicks $r\tau$)

avg. precip. rate ($\kappa/\tau$)



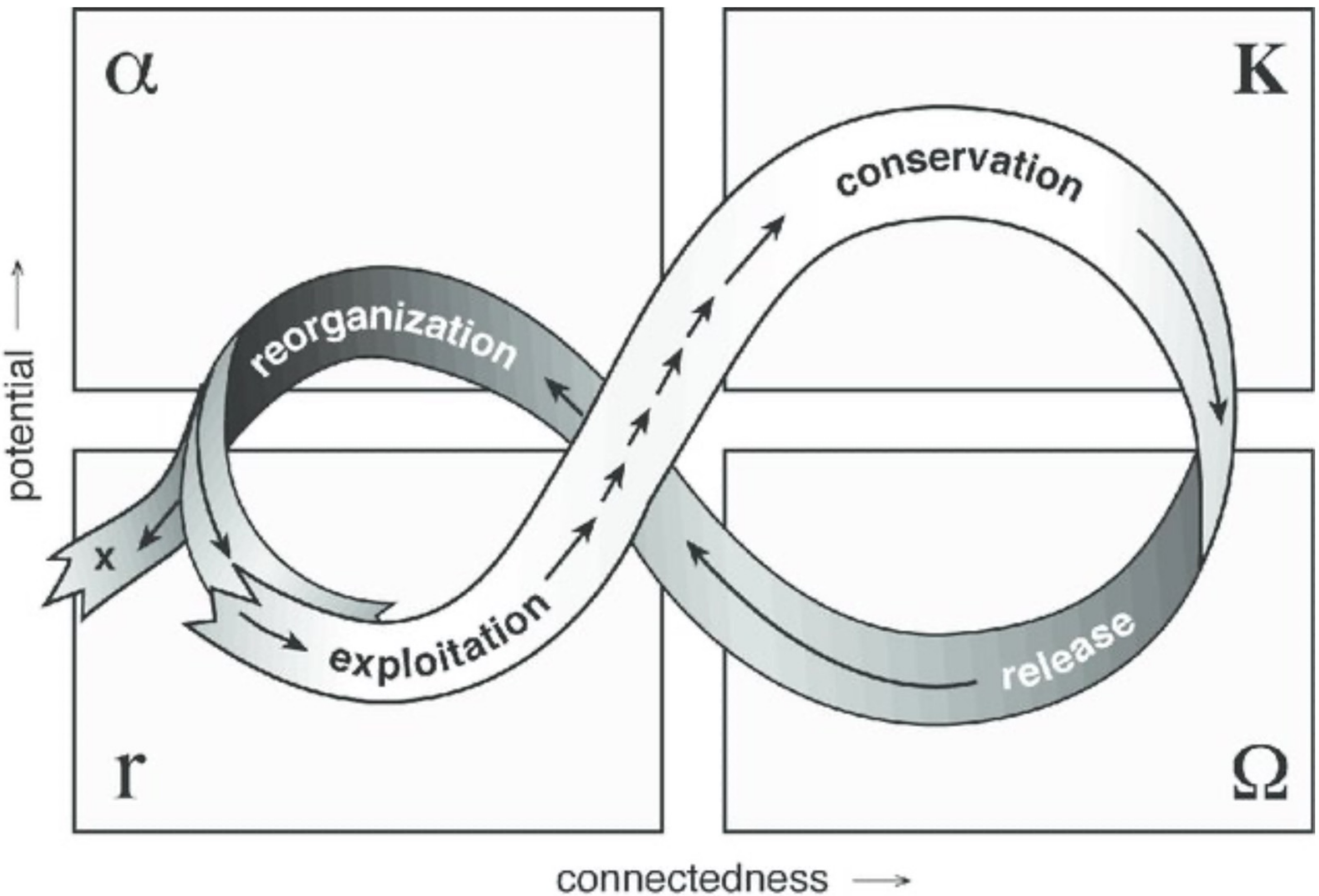
# Summary

- There is no single way to measure resilience---to gain clarity, ask "resilience of what to what?"

- Flow-kick models quantify resilience of attracting states (or desired states) to regular repeated, discrete disturbances

- Triggers for regime shifts can be unexpected:
  - maintain average disturbance rate but lower frequency
  - maintain kick size, but deliver less frequently

# Adaptive Cycle




The adaptive cycle (from Panarchy, edited by Lance H. Gunderson and C.S. Holling: Figure 2-1 (page 34). Copyright © 2002 Island Press.

https://www.resalliance.org/adaptive-cycle